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BOUNDARY VALUE AND EXPANSION PROBLEMS; FORMULATION OF VARIOUS TRANSCENDENTAL PROBLEMS.*

By R. D. Carmichael.

Proceeding under the guidance afforded by a previous investigation† of the algebraic basis of the theory of certain boundary value and expansion problems we here develop the formal aspects of several transcendental expansion problems which are rather different externally but are abstractly of much the same essence. In each case the work is carried forward to the point of derivation of the form of the expansion and the values of the coefficients; and the problem of determining the nature of the functions having expansions of each of the kinds enumerated is thus clearly set forth. It seemed desirable to present in a single paper this formulation of several problems, so that the reader may most conveniently compare their elements of similarity or difference; and to leave to further investigation the detailed solution of each problem deserving such further treatment. It will be apparent to every one that we have given merely a selection of problems from a much larger number which emerge in a similar way from the consideration of various sorts of limiting cases of the algebraic systems dealt with in the paper already mentioned.

In § 1 we treat the boundary value and expansion problem arising in connection with a single linear differential equation involving a single parameter. In § 2 the corresponding problem for a system of first order equations is more briefly treated and the results are carried over by Volterra's limiting process so as to yield the formulation of a similar expansion problem for certain integro-differential systems; and in § 3 the results of § 2 are rapidly generalized to problems involving ν parameters. Various problems involving difference and integro-difference equations are treated in § 4; these are analogous to the corresponding ones for differential and integro-differential equations developed in §§ 1, 2, 3. In § 5 is given a brief statement of a great variety of mixed problems involving one or more parameters. The character of the problems arising from partial differential equations and integro-differential equations is developed rapidly in § 6 and certain typical cases are briefly treated separately. A particular case of problems arising from partial differential equations is one which is connected with the theory of vibrating plates; this is briefly treated in § 7.

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† American Journal of Mathematics, Vol. XLIII (1921), p. 69.

More details are given in § 1 than elsewhere in the paper, the justification being that those which are omitted are fairly obvious to one who has in mind the corresponding developments in § 1.

1. The Problem Arising from a Single Differential Equation with One Parameter.

—In connection with the linear differential expressions

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u,$$

$$L_1(u) \equiv g_m \frac{d^m u}{dx^m} + g_{m-1} \frac{d^{m-1} u}{dx^{m-1}} + \cdots + g_1 \frac{du}{dx} + g_0 u, \quad m < n,$$

let us consider also the Lagrange adjoint expressions

$$M(v) \equiv (-1)^n \frac{d^n}{dx^n}[l_n v] + (-1)^{n-1} \frac{d^{n-1}}{dx^{n-1}}[l_{n-1} v] + \cdots - \frac{d}{dx}[l_1 v] + l_0 v$$

$$\equiv m_n \frac{d^n v}{dx^n} + \cdots + m_1 \frac{dv}{dx} + m_0 v,$$

$$M_1(v) \equiv h_m \frac{d^m v}{dx^m} + \cdots + h_1 \frac{dv}{dx} + h_0 v.$$

Here it is understood that m may be zero so that $L_1(u)$ may have the particular value $g_0 u$. We assume that the coefficients l_k, g_k, m_k, h_k, with their derivatives of all orders, are real-valued single-valued and continuous functions of x in an interval (ab) and that l_n and g_m do not vanish in (ab). Without loss of generality we take l_n to be positive in (ab). With the linear differential equation

$$(1) \qquad L(u) + \lambda L_1(u) = 0$$

and certain n linear homogeneous conditions in $u(a)$, $u'(a)$, $\cdots$, $u^{(n-1)}(a)$, $u(b)$, $u'(b)$, $\cdots$, $u^{(n-1)}b$,

$$(2) \qquad U_1(u) = 0, \qquad U_2(u) = 0, \qquad \cdots, \qquad U_n(u) = 0,$$

we associate the adjoint linear differential equation

$$(3) \qquad M(v) + \lambda M_1(v) = 0$$

and n like *adjoint conditions*

$$(4) \qquad V_{n+1}(v) = 0, \qquad V_{n+2}(v) = 0, \qquad \cdots, \qquad V_{2n}(v) = 0$$

in such wise as to generalize the corresponding problem treated by Birkhoff* for the case when $L_1(u) \equiv g_0 u$.

We must first make precise the boundary conditions. In doing this as well as in developing the other parts of this section we follow the suggestions afforded by Birkhoff's treatment of the special case. If we write

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* *Transactions of the American Mathematical Society*, 9 (1908), 373–395; 219–231.

$$R(u, v) \equiv [l_n v]\frac{d^{n-1}u}{dx^{n-1}} + \left[l_{n-1}v - \frac{d}{dx}(l_n v)\right]\frac{d^{n-2}u}{dx^{n-2}}$$
$$+ \left[l_{n-2}v - \frac{d}{dx}(l_{n-1}v) + \frac{d^2}{dx^2}(l_n v)\right]\frac{d^{n-3}u}{dx^{n-3}} + \cdots$$
$$+ \left[l_1 v - \frac{d}{dx}(l_2 v) + \frac{d^2}{dx^2}(l_3 v) - \cdots + (-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(l_n v)\right]u,$$
$$R_1(u, v)^* \equiv [g_m v]\frac{d^{m-1}u}{dx^{m-1}} + \cdots + \left[g_1 v - \frac{d}{dx}(g_2 v) + \frac{d^2}{dx^2}(g_3 v) - \cdots \right.$$
$$\left. + (-1)^{m-1}\frac{d^{m-1}}{dx^{m-1}}(g_m v)\right]u,$$

then it is well-known and easily verified that we have the identities

$$(5)\quad vL(u) - uM(v) \equiv \frac{d}{dx}\{R(u, v)\}, \qquad vL_1(u) - uM_1(v) \equiv \frac{d}{dx}\{R_1(u, v)\}.$$

Let us now consider the quantity

$$[R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b}$$

as a bilinear form in the two sets of variables $u^{(n-1)}(a), u^{(n-2)}(a), \cdots, u'(a), u(a), u^{(n-1)}(b), u^{(n-2)}(b), \cdots, u'(b), u(b)$ and $v(a), v'(a), \cdots, v^{(n-1)}(a), v(b), v'(b), \cdots, v^{(n-1)}(b)$, and let the form be arranged in a square array whose columns contain the u's in the order written and whose rows contain the v's in the order written. The matrix of this bilinear form obviously has the following properties: every element below the main diagonal is zero; every element in the upper right hand fourth of the matrix is zero, the division being made by horizontal and vertical lines through the center; the determinant of the matrix has the non-zero value $\{l_n(a)l_n(b)\}^n$.

Now this bilinear expression may be written in the normal form

$$(6)\qquad [R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b} \equiv \sum_{i=1}^{2n} U_i(u)V_i(v),$$

where the U_i and the V_i are linear and homogeneous functions with constant coefficients, the first in the variables u and the second in the variables v. From the totality of such representations we propose to choose those having certain properties.

In case $L_1(u) \equiv g_0 u$, so that our bilinear form is independent of λ, the linear forms U_i may be chosen in any way so that the $2n$ forms are linearly independent. The forms V_i are then uniquely determined and are linearly independent. This is the choice to be taken in the case of the Birkhoff problem.

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[*] If $m = 0$ we take $R_1(u, v)$ to be identically zero.

In the more general case when the bilinear form contains λ we wish to realize the following conditions: the linear forms $U_1, U_2, \cdots, U_n, V_{n+1}, V_{n+2}, \cdots, V_{2n}$ shall be independent of λ; the whole set $U_1, U_2, \cdots, U_{2n}$ shall be linearly independent for every value of λ; the whole set $V_1, V_2, \cdots, V_{2n}$ shall be linearly independent for every value of λ. For the last two conditions it is necessary and sufficient that the determinant of the set of linear forms U_i shall be independent of λ (except possibly for a factor dependent on λ but vanishing for no value of λ) and different from zero, as one sees readily by aid of the fact that the determinant of the U_i multiplied by the determinant of the V_i has a value equal to the (non-zero) value of the determinant of the bilinear form. That these three conditions may in fact be realized may be seen from the following particular choice of linear forms:

$$U_1 = u^{(n-1)}(a), \quad U_2 = u^{(n-2)}(a), \quad \cdots, U_{n-1} = u'(a), \ U_n = u^{(n-1)}(b);$$
$$V_{n+1} = v^{(n-1)}(a), \ V_{n+2} = v'(b), \quad V_{n+3} = v''(b), \ \cdots, \ V_{2n} = v^{(n-1)}(b);$$

for $i \leqq n$, $V_i =$ total factor of U_i in the fundamental bilinear form; for $i > n$, $U_i =$ total factor of V_i in the fundamental bilinear form. Indeed it is now evident that the needful properties may be realized in a great variety of ways.

When the forms U_i and V_i in (6) are determined in accordance with the conditions named, then equation (1) and conditions (2) set a given boundary value problem the *adjoint* of which is set by equation (3) and conditions (4). It is clear, moreover, that problem (1), (2) is likewise the adjoint of problem (3), (4), so that the relation between the two problems is a reciprocal one.

Remark 1.—We have seen that the conditions named can always be realized whatever the value of m, subject merely to the condition that it shall be less than n. For smaller values of m we also have certain additional freedom. We may more generally take k linear forms U_i and $2n - k$ linear forms V_i independent of λ where k is not necessarily equal to n but may be less or greater by an amount depending on m. When m is zero k may be any positive integer less than $2n$.* We may then make corresponding modifications in the two adjoint boundary problems. For the purposes of this paper, however, it is desirable to realize the maximum symmetry in the relations of the two problems; and consequently we confine our attention to the case indicated in the definition as explicitly stated above.

Remark 2.—It is proper to inquire whether the restriction that m shall be less than n is essential. The case $m > n$ reduces to the case $m < n$ by replacing λ by $1/\lambda$ (except for the particular value $\lambda = 0$), so that we need to consider only the case $m = n$. In this case the determinant of the bilinear form in the first member of (6) is not in general independent of λ; and hence the properties of the forms U_i and V_i could not be maintained in general, properties which are needed in the further development. These properties might, however, be retained in large measure when $m = n$ provided that g_n and its derivatives have suitable values at a and at b; but we shall not attempt to carry this case along with the other, owing to the special hypotheses which would be necessary and the complications which would raise in certain parts of the work.

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* See a treatment of this case by Bôcher, *Transactions of the American Mathematical Society,* 14 (1913), 403–420.

Remark 3.—The question of generalization to an equation of the form

$$L(u) + \lambda L_1(u) + \lambda^2 L_2(u) + \cdots + \lambda^k L_k(u) = 0$$

also arises. Under suitable hypotheses it is not difficult to maintain the necessary characteristics of the boundary conditions. Some of the theorems are still maintained without change;, but as early as theorem III certain difficulties would arise in general with this more extended case.

We may now readily prove the following theorem:

THEOREM I. *If for* $\lambda = \bar{\lambda}$ *a solution*[*] $\bar{u}(x)$ *of (1), (2) exists, then a solution* $\bar{v}(x)$ *of (3), (4) also exists for* $\lambda = \bar{\lambda}$; *if* $\bar{u}(x)$ *is unique (except for a constant factor),* $\bar{v}(x)$ *is also unique (except for a constant factor).*

Since $\bar{u}$ is a solution of (1), (2) for $\lambda = \bar{\lambda}$, we have

$$U_1(\bar{u}) = U_2(\bar{u}) = \cdots = U_n(\bar{u}) = 0,$$

and for some k

$$U_{n+k}(\bar{u}) \neq 0,$$

where in U_{n+k} we have the value $\bar{\lambda}$ of λ, since otherwise we should have

$$\bar{u}(a) = \bar{u}'(a) = \cdots = \bar{u}^{(n-1)}(a) = \bar{u}(b) = \bar{u}'(b) = \cdots = \bar{u}^{(n-1)}(b) = 0$$

and our function $\bar{u}(x)$ would be identically equal to zero, contrary to hypothesis.

In the n-fold linear spread of solutions $v(x)$ of (3) for $\lambda = \bar{\lambda}$, there is at least one, say $\bar{v}(x)$, which satisfies the $n - 1$ linear homogeneous conditions

$$V_{n+i}(\bar{v}) = 0,$$

where i runs over all the numbers of the set $1, 2, \cdots, n$ except k. Now we have $L(\bar{u}) + \bar{\lambda}L_1(\bar{u}) = 0$ and $M(\bar{v}) + \bar{\lambda}M_1(\bar{v}) = 0$. Hence if we multiply the second identity in (5) by $\bar{\lambda}$ and add to the first we find on integrating that $R(\bar{u}, \bar{v}) + \bar{\lambda}R_1(\bar{u}, \bar{v})$ is a constant. Therefore from (6) we see that $U_{n+k}(\bar{u})V_{n+k}(\bar{v}) = 0$; whence it follows that $V_{n+k}(\bar{v}) = 0$, and hence that all the conditions (4) are satisfied by $\bar{v}$, so that it is in fact a solution of (3), (4).

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[*] By "solution" we usually mean a function which satisfies the equation and conditions but is not identically equal to zero; when we intend otherwise we shall indicate that fact explicitly.

It remains to be shown that $\bar{v}$ is unique whenever $\bar{u}$ is unique. For this purpose let us suppose, if possible, that $\bar{v}$ is not unique. Then two linearly independent solutions of (3), (4) exist, say $\bar{v}$ and v^*. Then k and j exist such that

$$\begin{vmatrix} V_k(\bar{v}) & V_j(\bar{v}) \\ V_k(v^*) & V_j(v^*) \end{vmatrix} \neq 0;$$

for, otherwise, constants c and d would exist,* not both zero, such that $V_l(c\bar{v} + dv^*) = 0$ for $l = 1, 2, \cdots, n$, whence it would follow that the solution $v = c\bar{v} + dv^*$ would satisfy all the conditions $V_i(v) = 0$, $i = 1, 2, \cdots, 2n$, and this is impossible since the V's are linearly independent for every value of λ. Now choose u^*, linearly independent of $\bar{u}$, so as to satisfy the $n - 2$ conditions

$$U_i(u^*) = 0$$

for i running over the numbers of the set $1, 2, \cdots, n$, except k and j. Then from (5) and (6) we see that we have

$$\begin{aligned} U_k(u^*)V_k(\bar{v}) \ + U_j(u^*)V_j(\bar{v}) \ &= 0, \\ U_k(u^*)V_k(v^*) + U_j(u^*)V_j(v^*) &= 0. \end{aligned}$$

From this we see that $U_k(u^*) = 0 = U_j(u^*)$. Hence u^* satisfies (1), (2). Therefore, if the solution of (3), (4) is not unique for $\lambda = \bar{\lambda}$ neither is the solution of (1), (2) unique. Hence we conclude to the truth of the second part of the theorem.

A value of λ for which the system (1), (2) [and hence the system (3), (4)] has a solution will be called a *characteristic value*. The characteristic value is said to be *simple* if the solution corresponding to it is unique (except for a constant factor).

THEOREM II. *If* $y_1, y_2, \cdots, y_n$ *are n linearly independent solutions of* (*1*) *for* $\lambda = \bar{\lambda}$, *the condition that* $\bar{\lambda}$ *is a characteristic value is that the deter-*

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* In more detail one may proceed thus: distinct numbers k_1 and j_1 exist such that

$$V_{k_1}(\bar{v}) \neq 0, \qquad V_{j_1}(v^*) \neq 0.$$

If then

$$V_{k_1}(\bar{v})V_{j_1}(v^*) = V_{k_1}(v^*)V_{j_1}(\bar{v})$$

we have

$$V_{k_1}(v^*) \neq 0, \qquad V_{j_1}(\bar{v}) \neq 0.$$

But if our determinant is always zero it is sufficient to take

$$c = V_{k_1}(v^*), \qquad d = - V_{k_1}(\bar{v}),$$

in order to realize the result stated.

minant

$$\Delta = \begin{vmatrix} U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) \\ U_2(y_1) & U_2(y_2) & \cdots & U_2(y_n) \\ \cdot & \cdot & \cdot & \cdot \\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) \end{vmatrix}$$

shall vanish; a characteristic value λ *of* λ *is simple when and only when some first minor of* Δ *does not vanish.*

If we take the general solution of (1) in the form

$$u = c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

we see that the vanishing of Δ is a sufficient condition that conditions (2) may be satisfied through an appropriate choice of $c_1, c_2, \cdots, c_n$. When some first minor does not vanish this choice is unique (except for a constant factor).

THEOREM III. *If* $u_i(x)$ *and* $v_j(x)$ *are solutions of (1), (2) and (3), (4), respectively, the first for* $\lambda = \lambda_i$ *and the second for* $\lambda = \lambda_j$, *and if* $\lambda_i \neq \lambda_j$, *then we have*

$$\int_a^b u_iM_1(v_j)dx = 0, \qquad \int_a^b v_jL_1(u_i)dx = 0.$$

From the equations

$$L(u_i) + \lambda_iL_1(u_i) = 0, \qquad M(v_j) + \lambda_jM_1(v_j) = 0$$

we have

$$\{v_j[L(u_i) + \lambda_iL_1(u_i)] - u_i[M(v_j) + \lambda_iM_1(v_j)]\} + (\lambda_i - \lambda_j)u_iM_1(v_j) = 0.$$

If we integrate from a to b we see through aid of (2), (4), (6) that the expression inclosed in braces yields the value zero. Hence we have

$$(\lambda_i - \lambda_j)\int_a^b u_iM_1(v_j)dx = 0.$$

Dividing through by the factor $\lambda_i - \lambda_j$ we have the first relation in the theorem. The second may be proved in a similar way. Moreover, either of these relations follows readily from the other by means of the second relation in (5) and relation (6), looked upon as an identity in λ, together with the boundary conditions.

If we have two sets of functions $u_1(x)$, $u_2(x)$, $u_3(x)$, $\cdots$ and $v_1(x)$, $v_2(x)$, $v_3(x)$, $\cdots$, and the associated adjoint linear forms $L_1(u)$ and $M_1(v)$ such that relations (7) are satisfied for every i and j which are different we shall say that the two sets of functions are *adjoint* each to the other. If the two sets are the same and the form $L_1(u)$ is self-adjoint we shall say that the single set is *self-adjoint.* The conceptions thus introduced are obviously generalizations of the conceptions of biorthogonal sets and orthogonal sets of functions.

If now we have further

$$\int_a^b u_iM_1(v_i)dx \neq 0, \qquad i = 1, 2, 3, \cdots,$$

it is clear that we may formally determine the coefficients c_i in the formal expansion of a given function $f(x)$ in the form

$$f(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots.$$

In fact, if one multiplies by $M_1(v_i)$ and integrates from a to b one has formally

$$c_i = \frac{\int_a^b f(x) M_1(v_i)\,dx}{\int_a^b u_i M_1(v_i)\,dx}, \qquad i = 1, 2, 3, \cdots.$$

When λ is not a characteristic number for (1), (2) there exists* a unique $G(x, s; \lambda)$ such that the solution φ of

$$L(\varphi) + \lambda L_1(\varphi) = r, \qquad U_1(\varphi) = U_2(\varphi) = \cdots = U_n(\varphi) = 0,$$

is given by

$$\varphi = \int_a^b G(x, s; \lambda) r(s)\,ds;$$

and likewise there exists a unique $H(x, s; \lambda)$ such that the solution ψ of

$$M(\psi) + \lambda M_1(\psi) = r, \qquad V_{n+1}(\psi) = V_{n+2}(\psi) = \cdots = V_{2n}(\psi) = 0,$$

is given by

$$\psi = \int_a^b H(x, s; \lambda) r(s)\,ds.$$

Moreover, we have

$$G(x, s; \lambda) = H(s, x; \lambda).$$

Explicit formulae for G and H are readily obtained (see Bocher, l.c.). The formula for G may be written in terms of a fundamental system y_1, y_2. $\cdots$, y_n of solutions of (1) as follows:

$$(8) \quad G(x, s; \lambda) = \frac{N(x, s; \lambda)}{\Delta(\lambda)} = \frac{\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) & \bar{G}(x, s; \lambda) \\ U_1(y_1) & U_1(y_2) & \cdots & U_1(y_n) & U_1(\bar{G}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ U_n(y_1) & U_n(y_2) & \cdots & U_n(y_n) & U_n(\bar{G}) \end{vmatrix}}{\Delta(\lambda)}$$

where $\Delta(\lambda)$ has the value given in theorem II and $\bar{G}(x, s; \lambda)$ has the value

$$(9) \qquad \bar{G}(x, s; \lambda) = \pm \frac{1}{2} \frac{1}{l_n(s)} \frac{\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1^{(n-2)}(s) & y_2^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s) \\ \cdot & \cdot & \cdot & \cdot \\ y_1(s) & y_2(s) & \cdots & y_n(s) \end{vmatrix}}{\begin{vmatrix} y_1^{(n-1)}(s) & y_2^{(n-1)}(s) & \cdots & y_n^{(n-1)}(s) \\ y_1^{(n-2)}(s) & y_2^{(n-2)}(s) & \cdots & y_n^{(n-2)}(s) \\ \cdot & \cdot & \cdot & \cdot \\ y_1(s) & y_2(s) & \cdots & y_n(s) \end{vmatrix}},$$

the sign $+$ being taken when $x > s$ and the sign $-$ when $x < s$.

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* Bôcher, "Les Méthodes de Sturm," Chapter V.

Now the functions $y_1, y_2, \cdots, y_n$ may be taken analytic in λ. Hence G is analytic in λ except when $\Delta(\lambda) = 0$, that is, except for the characteristic values λ_i of λ for problem (1), (2). If λ_i is a simple characteristic value we may write

$$(10) \qquad G(x, s; \lambda) = \frac{R_i(x, s)}{\lambda - \lambda_i} + \sigma(x, s; \lambda)$$

where $\sigma(x, s; \lambda)$ is analytic in λ for $\lambda = \lambda_i$. From (8) we have

$$R_i(x, s) = \frac{N(x, s; \lambda_i)}{\Delta'(\lambda_i)},$$

where $\Delta'(\lambda_i)$ denotes the value at $\lambda = \lambda_i$ of the derivative with respect to λ of $\Delta(\lambda)$. In $N(x, s; \lambda_i)$ the coefficient of $\overline{G}$ is zero. Hence $N(x, s; \lambda_i)$ and its first n derivatives with respect to x are continuous. Moreover, it is easy to see that it satisfies (1), (2); therefore $R_i(x, s)$ is a solution of (1), (2). From the relation between G and H we infer that $R_i(x, s)$ is a solution of (3), (4) in s. From the definition of simple characteristic value it follows then that

$$R_i(x, s) = c_i u_i(x) v_i(s)$$

where c_i is independent of x and s and is yet to be determined.

Now from (10) we have

$$\lim_{\lambda=\lambda_i} \{(\lambda - \lambda_i) G(x, s; \lambda) - c_i u_i(x) v_i(s)\} = 0;$$

whence it follows that

$$(11) \quad \lim_{\lambda=\lambda_i} \left[(\lambda - \lambda_i) \int_a^b G(x, s; \lambda) L_1(u_i(s)) ds - c_i u_i(x) \int_a^b v_i(s) L_1(u_i(s)) ds\right] = 0.$$

But since

$$L(u_i) + \lambda L_1(u_i) = (\lambda - \lambda_i) L_1(u_i), \quad U_1(u_i) = U_2(u_i) = \cdots = U_n(u_i) = 0,$$

we have from the fundamental property of G that

$$(\lambda - \lambda_i) \int_a^b G(x, s; \lambda) L_1(u_i(s)) ds = u_i(x).$$

Substituting this value into equation (11) we have

$$c_i \int_a^b v_i(s) L_1(u_i(s)) ds = 1.$$

Similarly we should obtain the relation

$$c_i \int_a^b u_i(s) M_1(v_i(s)) ds = 1.$$

Hence we have for the residue $R_i(x, s)$ the value

$$(12) \qquad R_i(x, s) = \frac{u_i(x) v_i(s)}{\int_a^b v_i(s) L_1(u_i(s)) ds} = \frac{u_i(x) v_i(s)}{\int_a^b u_i(s) M_1(v_i(s)) ds}.$$

If Γ is a contour in the λ-plane which incloses (just) the characteristic numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$, and if the latter are simple, we have readily

$$\frac{1}{2\pi\sqrt{-1}}\int_\Gamma\int_a^b G(x, s; \lambda)\,\frac{M_i(v_i(s))}{v_i(s)}f(s)ds d\lambda = \sum_{i=1}^{n}\frac{\int_a^b f(s)M_1(v_i(s))ds}{\int_a^b u_i(s)M_1(v_i(s))ds}u_i(x). \tag{13}$$

Thus we have represented in the form of a contour integral the sum of a finite number of terms of the formal expansion

$$f(x) = \sum_{i=1}^{\infty}\frac{\int_a^b f(s)M_1(v_i(s))ds}{\int_a^b u_i(s)M_1(v_i(s))ds}u_i(x) \tag{14}$$

of a function $f(x)$ for the case when the characteristic numbers $\lambda_1, \lambda_2, \lambda_3, \cdots$ are all simple and $G(x, s; \lambda)$ has a pole of the first order. At a value λ_i of λ for which these conditions are not satisfied, the corresponding term in the formal expansion in (14) should be replaced by

$$\int_a^b R_i(x, s)\,\frac{M_1(v_i(s))}{v_i(s)}f(s)ds,$$

where $R_i(x, s)$ denotes the residue of $G(x, s; \lambda)$ at $\lambda = \lambda_i$ and $v_i(s)$ is the solution $N(x, s; \lambda_i)$ of (3), (4) in s. Thus we have a definite formal expansion for every given function $f(x)$ for which the integrals involved exist.

In accordance with the general plan of this paper we rest this investigation at this point. We have carried it far enough to set clearly the fundamental expansion problem associated with it, namely, the problem of determining those functions $f(x)$ for which the given formal expansions are valid. This problem Birkhoff (l.c.) has treated with great success for the special case considered in his paper. It appears likely that the methods of Birkhoff's paper are sufficient to carry through the more general investigation. The results obtained above are sufficient to initiate the further treatment.

In the following sections we derive formal expansions in connection with a great variety of boundary value and expansion problems. We do not go into as much detail as in this section; but the reader who has this section in mind will have no difficulty in seeing how the results actually obtained in each case set the fundamental expansion problem for that case.

2. **The Problem Arising from a System of Differential Equations of the First Order with One Parameter and from Related Integro-differential Equations.**—Let us consider the adjoint systems of differential equations

$$\frac{dy_i}{dx} = \sum_{j=1}^{n}(a_{ij} + \lambda\alpha_{ij})y_j, \qquad i = 1, 2, \cdots, n, \tag{1}$$

$$\frac{dz_i}{dx} = \sum_{j=1}^{n}(-a_{ji} - \lambda\alpha_{ji})z_j, \qquad i = 1, 2, \cdots, n. \tag{2}$$

If for fixed i we multiply these respectively by z_i and y_i, add the resulting equations member by member, sum as to i from 1 to n, and then integrate from a to b, we have

$$[y_1z_1 + y_2z_2 + \cdots + y_nz_n]_{x=a}^{x=b} = 0. \tag{3}$$

Let us now suppose that homogeneous linear boundary conditions (involving the end-points a and b) are set up on the y's and adjoint conditions on the z's so that (3), for the solutions to be considered, is satisfied in virtue of the boundary conditions alone. Moreover, let us suppose that this is done in such a way that the characteristic values of λ for (1) with its conditions are the same as those for (2) with its conditions, the situation being analogous to that treated in § 1, and that the number of characteristic values is infinite. Let us denote the distinct characteristic values and corresponding solutions by the symbols $\lambda^{(k)}$, $y_i^{(k)}$, $z_i^{(k)}$, $k = 1, 2, 3, \cdots$. In (1) let us now replace λ and y_i by $\lambda^{(k)}$ and $y_i^{(k)}$; in (2) let us replace λ and z_i by $\lambda^{(l)}$ and $z_i^{(l)}$, where $\lambda^{(l)} \neq \lambda^{(k)}$; then we have two new systems. For fixed i let us multiply the former by $z_i^{(l)}$ and the latter by $y_i^{(k)}$, add, sum as to i from 1 to n, integrate from a to b, and simplify the first member by means of the equation to which (3) reduces when y_i and z_i are replaced by $y_i^{(k)}$ and $z_i^{(l)}$. Thus we have

$$0 = \int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} [(a_{ij} + \lambda^{(k)}\alpha_{ij})y_j^{(k)}z_i^{(l)} - (a_{ji} + \lambda^{(l)}\alpha_{ji})y_i^{(k)}z_j^{(l)}]dx, \qquad k \neq l;$$

whence, on simplifying and dividing by $\lambda^{(k)} - \lambda^{(l)}$, we have

$$\int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ji}y_i^{(k)}z_j^{(l)}dx = 0, \qquad k \neq l. \tag{4}$$

If we suppose that the integral in the first member of (4) is different in value from zero when $k = l$, we have a convenient means for the formal determination of the coefficients c_k (independent of x and i) in the simultaneous formal expansions of n given functions $f_1(x), f_2(x), \cdots, f_n(x)$:

$$f_i(x) = \sum_{k=1}^{\infty} c_k y_i^{(k)}, \qquad i = 1, 2, \cdots, n. \tag{5}$$

For determining these constants c_k we proceed as follows: multiply both members of (5) by $\alpha_{ji}z_j^{(l)}$, sum as to i and as to j from 1 to n, and integrate from a to b; in this way we obtain a relation from which we have readily the value of c_l, namely:

$$c_l = \frac{\int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ji} f_i z_j^{(l)} dx}{\int_a^b \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ji} y_i^{(l)} z_j^{(l)} dx}, \qquad l = 1, 2, 3, \cdots. \tag{6}$$

Thus we have in (6) the formulæ for the coefficients c_k, *the same for each function* $f_i(x)$, needed in the joint expansions in (5) of the n functions $f_i(x)$. They are formulæ of great interest.

We have outlined very briefly the foregoing expansion problem, which is probably the same as that of an unpublished paper of Birkhoff (see abstract in *Bulletin of the American Mathematical Society,* (2) 25 (1919), p. 442), because we wish to utilize the suggestion afforded by it for setting up a related problem for certain integro-differential systems obtained from the foregoing differential systems by the limiting process which Volterra has so frequently employed and to such great advantage. It is clear that the notion of *joint expansion* utilized in connection with equation (5) is an essential element in the formulation of the problem. This notion I obtained not from Birkhoff's abstract (which is too brief for details of any sort) but from the dissertation of Dr. C. C. Camp which the author was kind enough to allow me to read in manuscript.* In this dissertation there is a detailed and full treatment of a very special case of the expansion problem formulated above; and the range of ideas developed has been of great use to me in the formulation of the boundary value and expansion problems of this paper.

If we apply to systems (1) and (2) the Volterra limiting process we may obtain the integro-differential equations

$$\frac{\partial y(x, s)}{\partial x} = \int_\alpha^\beta \{u(x, s, t) + \lambda v(x, s, t)\} y(x, t) dt, \tag{7}$$

$$\frac{\partial z(x, s)}{\partial x} = \int_\alpha^\beta \{- u(x, t, s) - \lambda v(x, t, s)\} z(x, t) dt, \tag{8}$$

where the range of variation of x is from a to b while that of s and t is from α to β. Let us multiply both members of (7) by $z(x, s)$ and both members of (8) by $y(x, s)$, add the two resulting equations member by member, and integrate with respect to s from α to β; then, on carrying out certain simplifications through an interchange of the order of integration with respect to s and t and through certain obvious reductions, we have formally the relation

$$\int_\alpha^\beta \frac{\partial}{\partial x} \{y(x, s) z(x, s)\} ds = 0.$$

Integrating with respect to x from a to b, we have

$$\int_\alpha^\beta \{y(b, s) z(b, s) - y(a, s) z(a, s)\} ds = 0, \tag{9}$$

a relation which corresponds to (3) for the differential systems.

---

* Since this was written a paper by A. Schur has appeared dealing with this problem; see *Mathematische Annalen* 82 (1921): 213–236. It contains (p. 216) a reference to a special case of the problem treated by Hilbert (Göttingen *Nachrichten*, 1906, pp. 474–480). This I had not seen before.

Dr. Camp's article will appear in the next volume of this JOURNAL. [Editor.]

The first member of the last equation is a sort of bilinear form in a non-enumerably infinite set of variables $y(a, s)$, $y(b, s)$, $z(a, s)$, $z(b, s)$, a set of four variables being formed for each s of the interval $(\alpha\beta)$. One desires to have a reduction of this to a normal form analogous to that used in connection with equation (6) of § 1. At present we do not attempt to solve this problem in general. What we desire is to have a set of boundary conditions on y and z separately, linear and homogeneous in character, so that these boundary conditions alone shall imply equation (9) and so that the y-problem and the z-problem shall have the same characteristic values λ infinite in number. Assuming that such equations have been set up, we seek to determine the formal character of the boundary value problem in its simplest aspects.

Let us represent characteristic numbers and corresponding solutions by λ_k, $y_k(x, s)$, $z_k(x, s)$, $k = 1, 2, 3, \cdots$. Let us replace λ and y in (7) by λ_k and y_k; let us replace λ and z in (8) by λ_l and z_l, $k \neq l$; let us multiply the first resulting equation member by member by $z_l(x, s)$ and the second by $y_k(x, s)$, add the two resulting equations member, integrate with respect to s from α to β and with respect to x from a to b, and simplify the resulting relation by aid of what equation (9) becomes when y and z are replaced by y_k and z_l respectively. From both members of the equation which so results remove the non-zero factor $\lambda_k - \lambda_l$; thus we have

$$(10) \qquad \int_a^b\int_\alpha^\beta\int_\alpha^\beta v(x, t, s)y_k(x, s)z_l(x, t)\,ds\,dt\,dx = 0, \qquad k \neq l.$$

If we suppose that the first member of (10) is different from zero for $k = l$ and if we seek constants c_k so that a given function $f(x, s)$ shall have the formal expansion

$$(11) \qquad f(x, s) = \sum_{k=1}^{\infty} c_k y_k(x, s)$$

we are led (through the relation obtained on multiplying both members of (11) by $v(x, t, s)z_l(x, t)$ and on integrating appropriately) to the formulæ for c_l, namely:

$$(12) \qquad c_l = \frac{\int_a^b\int_\alpha^\beta\int_\alpha^\beta v(x, t, s)f(x, s)z_l(x, t)\,ds\,dt\,dx}{\int_a^b\int_\alpha^\beta\int_\alpha^\beta v(x, t, s)y_l(x, s)z_l(x, t)\,ds\,dt\,dx}, \qquad l = 1, 2, 3, \cdots.$$

As a special case of the foregoing problem let us take that in which u and v are independent of x and let us write

$$u(x, s, t) = g(s, t), \qquad v(x, s, t) = h(s, t).$$

Solutions exist of the forms

$$y(x, s) = \rho(x)\sigma(s), \qquad z(x, s) = R(x)S(s),$$

where each function in the second members is a function of the single variable indicated. As special boundary conditions implying equation (9) we may take the following:

$$y(b, s) - y(a, s) = 0, \qquad z(b, s) - z(a, s) = 0.$$

These conditions now reduce to the following:

$$\rho(b) - \rho(a) = 0, \qquad R(b) - R(a) = 0.$$

If we substitute the foregoing special values of y and z into the special forms of (7) and (8) indicated we may separate variables in the way which is classic in the study of the partial differential equations of physics and so come through to the following relations:

$$\begin{aligned} \rho'(x) &= \mu\rho(x), \qquad \rho(b) - \rho(a) = 0, \\ \mu\sigma(s) &= \int_\alpha^\beta \{g(s, t) + \lambda h(s, t)\}\sigma(t)dt; \end{aligned} \tag{13}$$

$$\begin{aligned} R'(x) &= -\mu R(x), \qquad R(b) - R(a) = 0, \\ -\mu S(s) &= \int_\alpha^\beta \{g(s, t) + \lambda h(s, t)\}S(t)dt, \end{aligned} \tag{14}$$

where μ is a constant. In (13) and (14) we use the related constants μ and $-\mu$ so as to maintain the requisite property of adjointness. The differential system in each of these pairs of conditions is easily solved, appropriate characteristic values of μ being thus determined. Then the integral equation of the pair is to be solved, appropriate characteristic values of λ having been determined for each value of μ.

Let us further specialize. Suppose that we take $b = \pi$ and $a = -\pi$. Then for μ we have the values $n\sqrt{-1}$, where n is a positive or negative integer, and for $\rho(x)$ and $R(x)$ the corresponding values $e^{nx\sqrt{-1}}$ and $e^{-nx\sqrt{-1}}$ respectively. If we take $g(s, t) \equiv 0$ and employ a non-zero value of μ we have in the second line of (13) and in the second line of (14) an integral equation with parameter λ/μ in the one case and $-\lambda/\mu$ in the other case. It is well-known that the quantities α, β, $h(s, t)$ in this equation may be so chosen that for each value of μ corresponding values of λ exist giving rise to the functions $\cos ms$ for solutions where m runs over the set 1, 2, 3, $\cdots$. For the special case thus defined the expansion (11) has the form

$$f(x, s) = \sum_{m,\, n=1}^{\infty} (c_{mn} \cos ms e^{nix} + \gamma_{mn} \cos ms e^{-nix}).$$

If $f(x, s)$ is a real-valued function when x and s are real it is clear that c_{mn} and γ_{mn} must be conjugate imaginaries. Then the expansion reduces to the form

$$f(x, s) = \sum_{m,\, n=1}^{\infty} (a_{mn} \cos nx + b_{mn} \sin nx) \cos ms,$$

where a_{mn} and b_{mn} are real numbers. Furthermore, if $f(x, s)$ is an even function of x as well as of s we must have $b_{mn} = 0$ so that the expansion becomes the double cosine series

$$f(x, s) = \sum_{m,n=1}^{\infty} a_{mn} \cos ms \cos nx.$$

Thus we see that the expansions afforded by (11) and (12) contain as special cases some of the classical expansions of analysis.

The problems formulated in connection with (1), (2) and (7), (8) are capable of various generalizations. It will be sufficient to indicate briefly one of them. Let us consider the system of equations

$$(15) \quad \frac{\partial y_i(x, s)}{\partial x} = \sum_{j=1}^{n} (a_{ij} + \lambda\alpha_{ij}) y_j(x, s)$$

$$+ \int_\alpha^\beta \sum_{j=1}^{n} [\rho_{ij}(x, s, t) + \lambda\sigma_{ij}(x, s, t)] y_j(x, t)dt,$$

$$(16) \quad \frac{\partial z_i(x, s)}{\partial x} = \sum_{j=1}^{n} (- a_{ji} - \lambda\alpha_{ji}) z_j(x, s)$$

$$+ \int_\alpha^\beta \sum_{j=1}^{n} [- \rho_{ji}(x, t, s) - \lambda\sigma_{ji}(x, t, s)] z_j(x, t)dt,$$

for $i = 1, 2, \cdots, n$. Here in general the a_{ij} and α_{ij} are functions of x and s. In the special case when $\rho_{ij} \equiv 0 \equiv \sigma_{ij}$ and a_{ij} and α_{ij} are functions of x alone, we may take y_i and z_i to be functions of x alone, and we have our systems (1) and (2). It is clear that equations (7) and (8) are included in another way as a special case.

Let us multiply (15) by $z_i(x, s)$ and (16) by $y_i(x, s)$, integrate the resulting equations as to s from α to β and in the relation coming from (16) interchange the order of double integration and in the part of the formula containing this double integration interchange s and t, add the two equations thus gotten from (15) and (16), and in the resulting equation sum as to i from 1 to n. Finally in this resulting equation integrate as to x from a to b. Then we have the relation

$$(17) \quad \int_\alpha^\beta [\sum_{i=1}^{n} \{y_i(b, s)z_i(b, s) - y_i(a, s)z_i(a, s)\}]ds = 0.$$

This corresponds to our previous relation (9).

Let us suppose that linear homogeneous boundary conditions on the y_i and the z_i separately are so chosen that relation (17) is implied by these boundary conditions, that the y-problem and the z-problem have the same characteristic values λ infinite in number, and let us denote the corresponding values and solutions by λ_k, $y_i^{(k)}$, $z_i^{(k)}$, $k = 1, 2, 3, \cdots$. Then by a method of procedure which is now obvious we come through to the fundamental reciprocal relation

$$
(18)\quad \begin{aligned} &\int_a^b \Big\{ \int_\alpha^\beta \Big[\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ji}(x, s) y_i^{(k)}(x, s) z_j^{(l)}(x, s)\Big] ds \\ &\quad + \int_\alpha^\beta \int_\alpha^\beta \Big[\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ji}(x, t, s) y_i^{(k)}(x, s) z_j^{(l)}(x, t)\Big] ds dt \Big\} dx = 0, \quad k \neq l. \end{aligned}
$$

This is the generalized biorthogonality condition of which we have already found special cases.

Let us denote by D_{kl} the first member of (18) and let us suppose that $D_{kk} \neq 0$ for each k. Then if we have a set of expansions of the form

$$
(19)\qquad f_i(x, s) = \sum_{r=1}^{\infty} c_r y_i^{(r)}(x, s), \qquad i = 1, 2, \cdots, n,
$$

giving the simultaneous expansions of the n functions $f_i(x, s)$, $i = 1, 2, \cdots, n$, in terms of the $y_i^{(r)}(x, s)$, the coefficients c_r may be found by multiplying both sides of the equation

$$
\sum_{j=1}^{n} \alpha_{ji}(x, s) z_j^{(\nu)}(x, s) + \int_\alpha^\beta \Big[\sum_{j=1}^{n} \sigma_{ji}(x, t, s) z_j^{(\nu)}(x, t)\Big] dt,
$$

summing as to i from 1 to n, integrating as to s from α to β and as to x from a to b, and simplifying by use of (18); thus we have

$$
(20)\quad \begin{aligned} D_{\nu\nu} c_\nu = &\int_a^b \Big\{ \int_\alpha^\beta \Big[\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ji}(x, s) f_i(x, s) z_j^{(\nu)}(x, s) \\ &+ \int_\alpha^\beta \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ji}(x, t, s) z_j^{(\nu)}(x, t) f_i(x, s) dt\Big] ds \Big\} dx, \quad \nu = 1, 2, 3, \cdots. \end{aligned}
$$

It is clear that one may apply to (15) and (16) the Volterra limiting process and so replace these equations by others in which the discrete variable i has given way to an additional continuous variable, the resulting integro-differential equation containing a repeated integration, and that the formal expansion problem subsists in the new form. One might then employ a system of such equations and apply again to them the Volterra limiting process; and so on indefinitely.

3. **Differential and Integro-differential Problems Involving ν Parameters.**—Let us consider the ν systems of equations

$$
(1)\quad \frac{dy_{ki}}{dx} = \sum_{j=1}^{n_k} (a_{0kij} + \lambda_1 a_{1kij} + \lambda_2 a_{2kij} + \cdots + \lambda_\nu a_{\nu kij}) y_{kj}, \qquad i = 1, 2, \cdots, n_k,
$$

a separate system being formed for each value k of the set $1, 2, 3, \cdots, \nu$; and with these let us associate the ν systems of adjoint equations

$$
(2)\quad \frac{dz_{ki}}{dx} = \sum_{j=1}^{n_k} (- a_{0kji} - \lambda_1 a_{1kji} - \cdots - \lambda_\nu a_{\nu kji}) z_{kj}, \qquad i = 1, 2, \cdots, n_k.
$$

For fixed k and i multiply equation (1) member by member by z_{ki} and equation (2) member by member by y_{ki}, add the two resulting equations member by member, sum as to i from 1 to n_k, and integrate with respect to x from a to b; thus we have

$$(3) \qquad \left[y_{k1}z_{k1} + y_{k2}z_{k2} + \cdots + y_{kn_k}z_{kn_k}\right]_{x=a}^{x=b} = 0, \qquad k = 1, 2, \cdots, \nu.$$

Let us now suppose that for each value of k we set up on the quantities $y_{ki}(b)$, $y_{ki}(a)$, $i = 1, 2, \cdots, n_k$, and separately on the quantities $z_{ki}(b)$, $z_{ki}(a)$, $i = 1, 2, \cdots, n_k$, linear homogeneous boundary conditions (analogous to those developed in detail in § 1) of such sort that relations (3) are implied by the boundary conditions alone and that the y-problem and the z-problem have the same sets $\lambda_1^{(s)}, \lambda_2^{(s)}, \cdots, \lambda_\nu^{(s)}$ of characteristic values, that the number of sets is enumerably infinite and that these distinct sets are denoted by the named symbols for $s = 1, 2, 3, \cdots$. Let the corresponding solutions be denoted by $y_{ki}^{(s)}$, $z_{ki}^{(s)}$.

In equation (1) let us replace the quantities y and λ by corresponding quantities $y^{(s)}$ and $\lambda^{(s)}$; in equation (2) let us replace the quantities z and λ by corresponding quantities $z^{(r)}$ and $\lambda^{(r)}$ where $r \neq s$. For a fixed k multiply the first resulting equation member by member by $z_{ki}^{(r)}$ and the second by $y_{ki}^{(s)}$, add the two resulting equations member by member, sum as to i from 1 to n_k, integrate with respect to x from a to b, and simplify by what (3) becomes on replacing the quantities y and z by the corresponding quantities $y^{(s)}$ and $z^{(r)}$. In this way we are led to the following relations:

$$(4) \quad (\lambda_1^{(s)} - \lambda_1^{(r)})\int_a^b \sum_{i=1}^{n_k}\sum_{j=1}^{n_k} a_{1kji}y_{ki}^{(s)}z_{kj}^{(r)}dx + \cdots$$
$$+ (\lambda_\nu^{(s)} - \lambda_\nu^{(r)})\int_a^b \sum_{i=1}^{n_k}\sum_{j=1}^{n_k} a_{\nu kji}y_{ki}^{(s)}z_{kj}^{(r)}dx = 0,$$

a separate relation being formed for each value k of the set $1, 2, \cdots, \nu$. Now the differences $\lambda_1^{(s)} - \lambda_1^{(r)}, \cdots, \lambda_\nu^{(s)} - \lambda_\nu^{(r)}$ are not simultaneously zero; hence the determinant of their coefficients in the system (4) must have the value zero, that is, we must have

$$\left|\int_a^b \sum_{i=1}^{n_k}\sum_{j=1}^{n_k} a_{lkji}y_{ki}^{(s)}z_{kj}^{(r)}dx\right| = 0, \qquad r \neq s,$$

where the term of the determinant explicitly written is that in the lth column and kth row.

The last relation may be reduced to a more convenient form. In the kth row of the determinant let us replace the variable x of integration by t_k and the variables i and j of summation by i_k and j_k respectively. Then the relation reduces to

$$\int_a^b\int_a^b \cdots \int_a^b \left|\sum_{i_k=1}^{n_k}\sum_{j_k=1}^{n_k} a_{lkj_ki_k}(t_k)y_{ki_k}^{(s)}(t_k)z_{kj_k}^{(r)}(t_k)\right| dt_1dt_2 \cdots dt_\nu = 0, \qquad r \neq s,$$

the determinant employed being that whose element in lth column and kth row is that explicitly written. Then if we employ the symbol D to represent the determinant

$$(5) \qquad D_{i_1j_1 \cdots i_\nu j_\nu}(t_1, t_2, \cdots, t_\nu) = |a_{lkj_ki_k}(t_k)|$$

whose element in lth column and kth row is that explicitly written, we have for the final form of our generalized biorthogonality conditions the following:

$$(6) \qquad \begin{aligned} \int_a^b \int_a^b \cdots \int_a^b \Bigg\{ \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdots \sum_{i_\nu=1}^{n_\nu} \sum_{j_\nu=1}^{n_\nu} D_{i_1j_1 \cdots i_\nu j_\nu}(t_1, t_2, \cdots, t_\nu) \\ \times \prod_{h=1}^{\nu} y_{hi_h}^{(s)}(t_h) z_{h_{j_h}}^{(r)}(t_k) \Bigg\} dt_1 dt_2 \cdots dt_\nu = 0, \qquad r \neq s. \end{aligned}$$

In what follows we shall suppose that we have before us the case in which the first member of (6) is different from zero when $r = s$.

Now let us consider the set of functions

$$f_{i_1i_2 \cdots i_\nu}(t_1, t_2, \cdots, t_\nu), \qquad i_k = 1, 2, \cdots, n_k \text{ for } k = 1, 2, \cdots, \nu,$$

with regard to the question of their simultaneous expansion in the form

$$(7) \qquad f_{i_1i_2 \cdots i_\nu}(t_1, t_2, \cdots, t_\nu) = \sum_{s=1}^{\infty} c_s y_{1i_1}^{(s)}(t_1) y_{2i_2}^{(s)}(t_2) \cdots y_{\nu i_\nu}^{(s)}(t_\nu),$$

the coefficients c_s being the same for every function of the set. In case such an expansion exists we have a ready means of determining the coefficients c_s. The form of the first member of (6) suggests the operations to be performed. Let us denote by B_{rs} the first member of (6) and let us suppose that $B_{rr} \neq 0$ for $r = 1, 2, 3, \cdots$. Then for the determination of the c_r we have the relations

$$(8) \qquad \begin{aligned} B_{rr}c_r = \int_a^b \int_a^b \cdots \int_a^b \Bigg\{ \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_1} \cdot \\ \cdots \sum_{i_\nu=1}^{n_\nu} \sum_{j_1=1}^{n_\nu} D_{i_1j_1 \cdots i_\nu j_\nu}(t_1, t_2, \cdots, t_\nu) f_{i_1i_2 \cdots i_\nu}(t_1, t_2, \cdots, t_\nu) \\ \cdot z_{1i_1}^{(r)}(t_1) z_{2i_2}^{(r)}(t_2) \cdots z_{\nu i_\nu}^{(r)}(t_\nu) \Bigg\} dt_1 dt_2 \cdots dt_\nu, \qquad r = 1, 2, 3, \cdots. \end{aligned}$$

The expansion problem which is set by these formal results we may look upon as a generalization of that briefly treated at the beginning of § 2. From the fuller treatment of a related problem given in § 1 one sees readily how to proceed to a more precise formulation of the problem here briefly characterized. Moreover, it is clear that one may formulate the problem at once for a system of ν equations of general orders and so include in one form (rather complicated, to be sure) the three differential equations problems already set.

We saw in § 12 how the integro-differential equations problem (7), (8) grew out of the differential equations problem (1), (2). In a similar way the problem (1), (2) of this section gives rise to a new problem associated with the equations

$$(9)\qquad \frac{\partial y_k(x, s)}{\partial x} = \int_\alpha^\beta \{a_{0k}(x, s, t) + \lambda_1 a_{1k}(x, s, t) + \cdots + \lambda_\nu a_{\nu k}(x, s, t)\} y_k(x, t)dt,$$

$$(10)\qquad \frac{\partial z_k(x, s)}{\partial x} = \int_\alpha^\beta \{- a_{0k}(x, t, s) - \lambda_1 a_{1k}(x, t, s) - \cdots - \lambda_\nu a_{\nu k}(x, t, s)\} z_k(x, t)dt.$$

With the guide afforded by the earlier part of this section and the special case of our present problem treated in § 2 one may proceed readily to set up the generalized biorthogonality conditions and by aid of them to determine the coefficients in the formal expansion of a function of 2ν variables.

Corresponding to equation (6) we should thus have

$$(11)\qquad \int_a^b\int_a^b\cdots\int_a^b \{\int_\alpha^\beta\int_\alpha^\beta\cdots\int_\alpha^\beta D(x_1, x_2, \cdots, x_\nu, s_1, t_1, s_2, t_2, \cdots, s_\nu, t_\nu) \cdot \prod_{h=1}^{\nu} y_h^{(s)}(x_h, s_h) z_h^{(r)}(x_h, t_h) ds_1 dt_1 \cdots ds_\nu dt_\nu\} dx_1 dx_2 \cdots dx_\nu = 0, \qquad r \neq s,$$

where the symbol D denotes the determinant of order ν whose element in lth column and kth row is $a_{lk}(x_k, s_k, t_k)$. If we denote the first member of this relation by C_{rs} and suppose that $C_{rr} \neq 0$ for every r, then for the coefficients c_r in the expansion

$$(12)\qquad f(x_1, x_2, \cdots, x_\nu, s_1, s_2, \cdots, s_\nu) = \sum_{s=1}^{\infty} c_s \prod_{h=1}^{\nu} y_h{}^{(s)}(x_h, s_h)$$

we have formally

$$(13)\qquad C_{rr}c_r = \int_a^b\int_a^b\cdots\int_a^b \{\int_\alpha^\beta\int_\alpha^\beta\cdots\int_\alpha^\beta D(x_1, \cdots, x_\nu, s_1, t_1, \cdots, s_\nu, t_\nu) f(x_1, \cdots, x_\nu, s_1, \cdots, s_\nu) \cdot \prod_{h=1}^{\nu} z_h^{(r)}(x_h, t_h) \cdot ds_1 dt_1 \cdots ds_\nu dt_\nu\} \times dx_1 dx_2 \cdots dx_\nu, \qquad r = 1, 2, 3, \cdots.$$

These results might, if it were desired, be also extended by means of the suggestion afforded by the problem associated with equations (15) and (16) of § 2 and the generalizations of it which are mentioned there.

4. **Problems Involving Difference and Integro-difference Equations.**— Let us consider the adjoint systems of difference equations

$$(1)\qquad u_i(x + 1) = \sum_{j=1}^{n} (\delta_{ij} + \varphi_{ij} + \lambda\psi_{ij}) u_j(x), \qquad i = 1, 2, \cdots, n,$$

$$(2)\qquad v_i(x) = \sum_{j=1}^{n} (\delta_{ji} + \varphi_{ji} + \lambda\psi_{ji}) v_j(x + 1), \quad i = 1, 2, \cdots, n,$$

where δ_{ij} has the value 1 or 0 according as i is or is not equal to j and φ_{ij}

and ψ_{ij} are functions of x which are analytic at infinity and vanish there to an order at least as high as the second. Under such hypotheses these equations have for a fixed value of λ fundamental systems*

$$u_{1j}(x),\ u_{2j}(x),\ \cdots,\ u_{nj}(x);\ v_{1j}(x),\ v_{2j}(x),\ \cdots,\ v_{nj}(x)$$

of solutions each function of which is analytic throughout the finite plane except for a set of singularities in a left half-plane; moreover, if x approaches infinity along any ray from the origin exclusive of the negative axis of reals or along any line proceeding to the right parallel to the axis of reals, we have

$$\lim u_{ij}(x) = \delta_{ij}, \qquad \lim v_{ij}(x) = \delta_{ij}.$$

Any solution of either equation formed from the foregoing solutions of that equation by taking a linear combination of such solutions with constant coefficients will have the property that each function in it approaches a limiting value as x approaches infinity in the manner specified. We confine attention to such solutions of equations (1) and (2).

Now if we multiply equation (1) member by member by $v_i(x + 1)$ and (2) by $- u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$\sum_{i=1}^{n}\Delta\{u_i(x)v_i(x)\} = 0. \tag{3}$$

If the real part of a is sufficiently large we have u_i and v_i analytic at every point x whose real part is not less than the real part of a. Hence in (3) we may sum as to x from a to infinity, where x runs over the values a, $a + 1$, $a + 2$, $\cdots$; thus we have

$$\sum_{i=1}^{n}\{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} = 0. \tag{4}$$

This is clearly analogous to equation (3) of § 2 and differs from it (apart from notation) only in having the limit ∞ instead of the limit b.

Let us now suppose that adjoint homogeneous linear boundary conditions are set up on the $u_i(\infty)$, $u_i(a)$ on the one hand and on the $v_i(\infty)$, $v_i(a)$ on the other hand so that relation (4) is implied by the boundary conditions and that this is done in such a way that the characteristic values of λ for (1) and the conditions on the u_i are the same as those for (2) and the conditions on the v_i and that the number of characteristic values is infinite. Let us denote the distinct characteristic values and corresponding solutions by $\lambda^{(k)}$, $u_i^{(k)}$, $v_i^{(k)}$, $k = 1, 2, 3, \cdots$.

In (1) let us now replace λ and u_i by $\lambda^{(k)}$ and $u_i^{(k)}$, in (2) let us replace λ and v_i by $\lambda^{(l)}$ and $v_i^{(l)}$, where $l \neq k$; then we have two new systems. Let us multiply the first resulting equation member by member by $v_i^{(l)}(x+1)$ and the second by $-u_i^{(k)}(x)$, add the resulting equations member by member, sum as to i from 1 to n, sum as to x over the set $a, a+1, a+2, \cdots$, simplify by aid of what (4) becomes when u_i and v_i are replaced by $u_i^{(k)}$ and $v_i^{(l)}$ respectively, and divide by the non-zero quantity $\lambda^{(k)} - \lambda^{(l)}$. Thus we have

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* See AMERICAN JOURNAL OF MATHEMATICS, 35 (1913), 161–162.

$$(5) \qquad \sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(a+t)u_i^{(k)}(a+t)v_j^{(l)}(a+1+t) = 0, \quad k \neq l.$$

If we suppose that the quantity in the first member of (5) is (for all k) different from zero when l is replaced by k we shall have a ready means for the formal determination of the coefficients c_k in the (assumed) expansions

$$(6) \qquad f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \quad i = 1, 2, \cdots, n,$$

the same coefficients c_k being employed for each of the n functions. For this purpose we multiply both sides of equation (6) by $\psi_{ji}(x)v_j^{(k)}(x+1)$, sum as to i and j each from 1 to n, and sum as to x over the set $a, a+1, a+2, \cdots$; employing (5) we come through readily to the relations

$$(7) \qquad c_k = \frac{\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(a+t)f_i(a+t)v_j^{(k)}(a+1+t)}{\sum_{t=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_{ji}(a+t)u_i^{(k)}(a+t)v_j^{(k)}(a+1+t)}, \qquad k = 1, 2, 3, \cdots.$$

Since the coefficients c_k are determined merely by the function-values at a discrete set of values of x it may seem at first sight that this is not likely to lead to an expansion theory of much interest. But it would be natural to confine attention to functions $f_i(x)$ having suitably restricted properties, especially in the neighborhood of infinity, and for such functions it may be anticipated that the theory is valuable. Moreover, it furnishes also a natural means of interpolation of a function defined initially only over the set of points $a, a+1, a+2, \cdots$, the function values at these points being the only ones needed in the determination of the coefficients c_k. Again, it affords a natural means of formal expansion of functions defined only for positive integral values of the argument, such as is the case with the usual number-theoretic functions.

Let us rapidly illustrate the foregoing theory by considering the problem associated with the equations

$$(8) \qquad u(x+1) - \left(1 - \frac{\mu^2}{x^2}\right)u(x) = 0,$$

$$(9) \qquad \left(1 - \frac{\mu^2}{x^2}\right)v(x+1) - v(x) = 0,$$

where μ^2 is the parameter. These equations have the solutions

$$(10)\qquad u(x)=\frac{\Gamma(x-\mu)\Gamma(x+\mu)}{[\Gamma(x)]^2},\qquad v(x)=\frac{[\Gamma(x)]^2}{\Gamma(x-\mu)\Gamma(x+\mu)},$$

where $\Gamma(x)$ denotes the usual gamma function; and the general solution for each case is gotten by multiplying the particular solution by an arbitrary periodic function of period unity. But if $u(x)[v(x)]$ is to approach a definite limiting value when x approaches infinity in the sense defined above this periodic multiplier must reduce to a constant; and such constant multiplier is clearly irrelevant to the problem in hand. Hence, for our purposes, we take the solutions given in (10).

Now condition (4) reduces in the present case to $u(\infty)v(\infty)-u(a)v(a)=0$. As suitable boundary conditions implying this relation we may take

$$(11)\qquad t\,u(\infty)-u(a)=0,\qquad \frac{1}{t}v(\infty)-v(a)=0,$$

where t is a given non-zero constant. Now from (10) and the asymptotic properties of the gamma function we see that $u(\infty)=1=v(\infty)$. Then the condition of consistency for the u-problem and that for the v-problem both reduce to the following:

$$(12)\qquad \Gamma(a-\mu)\Gamma(a+\mu)=t\{\Gamma(a)\}^2.$$

The roots of this equation give the common characteristic values of μ^2 for the u-problem and the v-problem. [For each solution μ of (12) there is also the solution $-\mu$; but only their common square μ^2 is a characteristic value for our problem.]

For varying choice of a and t the expansion problem which results brings a variety of properties to notice. One of the most interesting cases is that in which $a=\frac{1}{2}$, $-1\leqq 1/t\leqq +1$; in this case equation (12) readily becomes

$$(13)\qquad \cos\pi\mu=\frac{1}{t},$$

if one reduces by aid of the relation $\Gamma(s)\Gamma(1-s)=\pi/\sin\pi s$. Then the characteristic values μ^2 are readily determined in terms of a single one of them. Let $\bar{\mu}$ be the smallest positive value of μ satisfying (13); then the characteristic values μ^2 are

$$\mu^2=(\bar{\mu}+2n)^2,\qquad n=0,\pm 1,\pm 2,\pm 3,\cdots.$$

Corresponding to these let us denote the solutions by $u_n(x)$, $v_n(x)$ so that we have*

$$(14) \qquad u_n(x) = \frac{\Gamma(x - \bar{\mu} - 2n)\Gamma(x + \bar{\mu} + 2n)}{[\Gamma(x)]^2}, \qquad v_n(x) = \frac{1}{u_n(x)}, \qquad n = 0, 1, 2, \cdots.$$

The biorthogonality conditions now reduce to

$$(15) \qquad \sum_{k=0}^{\infty} \frac{1}{(k+\frac{1}{2})^2} \frac{u_n(k+\frac{1}{2})}{u_m(k+\frac{3}{2})} = 0, \qquad m \neq n.$$

Now we may apply to systems (1) and (2) a Volterra limiting process analogous to that by which we arrived at equations (7) and (8) in § 2. By so doing we are led to the integro-difference equations †

$$(16) \qquad \Delta u(x, s) = \int_a^\beta \{\varphi(x, s, t) + \lambda\psi(x, s, t)\} u(x, t)dt,$$

$$(17) \qquad -\Delta v(x, s) = \int_a^\beta \{\varphi(x, t, s) + \lambda\psi(x, t, s)\} v(x + 1, t)dt.$$

Corresponding to equations (4) to (7) we now have the following:

$$(18) \qquad \int_a^\beta \{u(\infty, s)v(\infty, s) - u(a, s)v(a, s)\} ds = 0,$$

$$(19) \qquad \sum_{r=0}^{\infty} \int_a^\beta \int_a^\beta \psi(a + r, t, s) u_k(a + r, s) v_l(a + 1 + r, t) ds dt = 0, \qquad k \neq l,$$

$$(20) \qquad f(x, s) = \sum_{k=1}^{\infty} c_k u_k(x, s),$$

$$(21) \qquad c_k = \frac{\sum_{r=0}^{\infty} \int_a^\beta \int_a^\beta \psi(a + r, t, s) f(a + r, s) v_k(a + 1 + r, t) ds dt}{\sum_{r=0}^{\infty} \int_a^\beta \int_a^\beta \psi(a + r, t, s) u_k(a + r, s) v_k(a + 1 + r, t) ds dt}, \qquad k = 1, 2, 3, \cdots.$$

The foregoing work of this section is analogous to a part of that of § 2. It is clear that the analogues of the other parts of § 2 exist here and can be readily developed. In fact, we can develop the theory of difference and integro-difference equations in forms analogous to all parts of the problems treated or suggested in §§ 1 to 3 for differential and integro-differential equations.

5. **Mixed Problems Involving One or More Parameters.**—In view of the basic algebraic theory and the transcendental problems treated in §§ 1–4, it is clear that there must exist in the theory of integral equations expansion problems involving n parameters not only in the classic case when $n = 1$ but also in the general case when n is any positive integer. It was my intention to exhibit the basic formulæ on which such an expansion theory may be based; but, while this paper was in course of preparation, an abstract

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* It should be observed that the negative values of n merely give us the same solutions over again. It is also worth noting that $u_n(x)/u_1(x)$ is a rational function of x so that from the expansion of $f(x)$ in terms of $u_n(x)$ we have that of $f(x)/u_1(x)$ in terms of a certain interesting set of rational functions.

† Here $\Delta f(x, s) = f(x + 1, s) - f(x, s)$.

of a paper by Mrs. A. J. Pell appeared in the *Bulletin of the American Mathematical Society*, Ser. 2, Vol. 26 (1920), p. 149, indicating the character of this expansion problem for the case of two parameters. Consequently we shall leave out the formulæ for this case.

If we think of the several types of expansion problems—those for differential, difference, and integral equations—in intimate connection with the basic algebraic theory (developed in the memoir already referred to), it becomes apparent at once that the case of n parameters (for $n > 1$) is not confined to a set of n equations of the same sort. There is nothing to prevent one subset of the basic algebraic equations proceeding to differential equations as limiting forms, another to difference equations, another to integral equations, and so on. Thus we can see beforehand that we may formulate the expansion problem for a variety of mixed systems. In fact, we were led incidentally in § 2 to a special case of such mixed problems.

Let us now indicate more precisely the nature of these mixed problems by considering the equations

$$(1) \qquad \frac{dy_i}{dx} = \sum_{j=1}^{n} (a_{ij} + \lambda b_{ij} + \mu c_{ij} + \nu d_{ij}) y_j, \quad i = 1, 2, \cdots, n,$$

$$(2) \quad \Delta s_i(x) = \sum_{j=1}^{m} (\varphi_{ij} + \lambda \rho_{ij} + \mu \sigma_{ij} + \nu \tau_{ij}) s_j(x), \quad i = 1, 2, \cdots, m,$$

$$(3) \qquad u(x) = \lambda \int_a^b K(x, \xi) u(\xi) d\xi + \mu \int_a^b L(x, \xi) u(\xi) d\xi + \nu \int_a^b M\ (x, \xi) u(x, \xi) d\xi,$$

associating with them their adjoint equations

$$(4) \qquad \frac{dz_i}{dx} = \sum_{j=1}^{n} (- a_{ji} - \lambda b_{ji} - \mu c_{ji} - \nu d_{ji}) z_j, \quad i = 1, 2, \cdots, n,$$

$$(5) \quad \Delta t_i(x) = \sum_{j=1}^{m} (- \varphi_{ji} - \lambda \rho_{ji} - \mu \sigma_{ji} - \nu \tau_{ji}) t_j(x + 1), \quad i = 1, 2, \cdots, m,$$

$$(6) \qquad v(x) = \lambda \int_a^b K(\xi, x) v(\xi) d\xi + \mu \int_a^b L(\xi, x) v(\xi) d\xi + \nu \int_a^b M(\xi, x) v(\xi) d\xi.$$

We suppose that the φ, ρ, σ, τ functions are analytic at infinity and vanish there to at least the second order.

From (1) and (4), from (2) and (5), by methods already frequently illustrated, we have

$$(7) \qquad \left[y_1 z_1 + y_2 z_2 + \cdots + y_n z_n \right]_{x=a'}^{x=b'} = 0,$$

$$(8) \qquad \sum_{i=1}^{m} \left[s_i(\infty) t_i(\infty) - s_i(\alpha) t_i(\alpha) \right] = 0.$$

We are now to set up homogeneous linear boundary conditions on y and on z so that (7) is implied by them; and on s and on t so that (8) is implied by them. We shall suppose that this is done in such a way that the sets of characteristic values for the two problems [(1), (2), (3) on the one hand

and (4), (5), (6) on the other] are the same and that the number of the sets is enumerably infinite. Then we shall denote the corresponding values and solutions by

$$\lambda^{(k)},\ \mu^{(k)},\ \nu^{(k)},\ y^{(k)},\ z^{(k)},\ s_i^{(k)},\ t_i^{(k)},\ u^{(k)},\ v^{(k)},\quad k = 1, 2, 3, \cdots.$$

In equations (1), (2), (3) let us replace the symbols for the parameters and the functions sought by the corresponding symbols with the superscript k; in equations (4), (5), (6) let us introduce similarly the distinct superscript l; and let us denote in order the six new equations so formed by $(\bar{1})$, $(\bar{2})$, $\cdots$, $(\bar{6})$. From $(\bar{1})$, $(\bar{4})$ and the boundary conditions implying (7) we have readily

$$\begin{aligned}(\lambda^{(k)} - \lambda^{(l)})\int_a^b \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ji}y_i^{(k)}z_j^{(l)}dx + (\mu^{(k)} - \mu^{(l)})\int_a^b \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ji}y_i^{(k)}z_j^{(l)}dx \\ + (\nu^{(k)} - \nu^{(l)})\int_a^b \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ji}y_i^{(k)}z_j^{(l)}dx = 0.\end{aligned} \tag{9}$$

Similarly, from $(\bar{2})$, $(\bar{5})$ and the boundary conditions implying (8), we have

$$\begin{aligned}(\lambda^{(k)} - \lambda^{(l)})\sum_{w=0}^{\infty}\sum_{i=1}^{m}\sum_{j=1}^{m} \rho_{ji}(\alpha + w)s_i(\alpha + w)t_j(\alpha + 1 + w) + \cdots \\ + (\nu^{(k)} - \nu^{(l)})\sum_{w=0}^{\infty}\sum_{i=1}^{m}\sum_{j=1}^{m} \tau_{ji}(\alpha + w)s_i(\alpha + w)t_j(\alpha + 1 + w) = 0.\end{aligned} \tag{10}$$

From $(\bar{3})$ and $(\bar{6})$ we get likewise

$$\begin{aligned}(\lambda^{(k)} - \lambda^{(l)})\int_a^b\int_a^b K(\xi, x)u^{(k)}(x)v^{(l)}(\xi)dxd\xi + \cdots \\ + (\nu^{(k)} - \nu^{(l)})\int_a^b\int_a^b M(\xi, x)u^{(k)}(x)v^{(l)}(\xi)dxd\xi = 0.\end{aligned} \tag{11}$$

Since the parametric differences $\lambda^{(k)} - \lambda^{(l)}$, $\cdots$, involved in the last three equations are not simultaneously zero when $k \neq l$ the determinant of the coefficients of these differences must have the value zero when $k \neq l$. The relations thus arising express the generalized biorthogonality property of the solutions pertaining to the problem in hand. Once these are obtained the formal determination of the coefficients in the formal expansions

$$f_{ij}(x_1, x_2, x_3) = \sum_{k=1}^{\infty} c_k y_i^{(k)}(x_1)s_j^{(k)}(x_2)u^{(k)}(x_3), \qquad i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, m, \tag{12}$$

of a set of mn functions of three variables is easily effected by the method already employed several times in this paper.

The foregoing we have outlined briefly as an instance of a sort of thing which may be done in a great variety of ways. The formal aspects of the problem are easily developed whether we have a single equation of a given type, as in the problem here treated, or any number of equations of each type. Also, we may employ in the list of types integro-difference and integro-differential equations. In view of the developments of the section following it will be seen that partial differential and partial difference equations are also admissible. Similarly one may allow equations involving functions of two variables and the operation of differentiation with respect to one and that of differencing with respect to the other; and so on. In this vast totality most types are without special interest; but a few actually arise in problems which come naturally to our attention.

Nor is this all. It is also possible to treat similarly equations in which functions are subjected to different operations with respect to the same variable. We illustrate this with one of the simplest cases. Let D_i denote a set of n operations, the first k of them being identical with D and the remaining ones with Δ, where k is some one of the numbers 0, 1, $\cdots$, n, and where D is the symbol for differentiation and Δ is the symbol for the operation $\Delta f(x) = f(x+1) - f(x)$. Then form the system of equations

$$D_i u_i(x) = \sum_{j=1}^{n} (a_{ij} + \lambda b_{ij}) u_j(x), \quad i = 1, 2, \cdots, n, \tag{13}$$

and associate with them the adjoint system

$$D_i v_i(x) = \sum_{j=1}^{n} (-a_{ji} - \lambda b_{ji}) v_j(x + \epsilon_j), \quad i = 1, 2, \cdots, n, \tag{14}$$

where ϵ_i is zero or unity according as D_i is D or Δ. If we multiply (1) by $v_i(x + \epsilon_i)$ and (2) by $u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have a result which may be put in the form

$$\sum_{i=1}^{n} D_i \{u_i(x) v_i(x)\} = 0. \tag{15}$$

Then, if appropriate conditions at infinity are satisfied, we have

$$\sum_{i=1}^{n} \{u_i(\infty) v_i(\infty) - u_i(a) v_i(a)\} = 0.$$

It is clear now how one may complete the formulation of the expansion problem; the work proceeds in the closest analogy with special cases of it in §§ 2 and 4.

6. **Problems Arising from Partial Differential Equations and Integro-differential Equations.**—As typical of the general situation in regard to expansion problems arising from the theory of partial differential equations, let us consider that which is to be associated with the equation

$$L(u) + \lambda L_1(u) = 0 \tag{1}$$

and its adjoint

$$M(v) + \lambda M_1(v) = 0,$$

where

$$L(u) \equiv a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\partial y} + c\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + \gamma u,$$

$$L_1(u) \equiv \beta\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} + ru,$$

$$M(v) \equiv \frac{\partial^2}{\partial x^2}(av) + 2\frac{\partial^2}{\partial x\partial y}(bv) + \frac{\partial^2}{\partial y^2}(cv) - \frac{\partial}{\partial x}(\alpha v) - \frac{\partial}{\partial y}(\beta v) + \gamma v,$$

$$M_1(v) \equiv -\frac{\partial}{\partial x}(pv) - \frac{\partial}{\partial y}(qv) + ru,$$

where the symbols u, v, a, b, c, α, β, γ, p, q, r denote functions of the independent variables x and y.

Let us multiply both members of equation (1) by v and of equation (2) by $-u$ and add the resulting equations member by member; thus we have

$$\begin{aligned}&\frac{\partial}{\partial x}\left[a\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + b\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(\alpha + \lambda p - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right)uv\right]\\ &\qquad + \frac{\partial}{\partial y}\left[c\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + b\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial y}\right)\right.\\ &\qquad\qquad \left. + \left(\beta + \lambda q - \frac{\partial c}{\partial y} - \frac{\partial b}{\partial x}\right)uv\right] = 0.\end{aligned} \tag{3}$$

Let S be any closed region in the xy-plane whose boundary is cut twice and only twice by a line through it and parallel to either axis. Then if we integrate over S both members of equation (3) we have a relation which reduces readily to the form

$$\begin{aligned}&\int_\rho^\sigma\left[a\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right) + b\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + \left(\alpha + \lambda p - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right)uv\right]_{x=r_y}^{x=s_y} dy\\ &\qquad + \int_r^s\left[c\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}\right) + b\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}\right)\right.\\ &\qquad\qquad \left. + \left(\beta + \lambda q - \frac{\partial c}{\partial y} - \frac{\partial b}{\partial x}\right)uv\right]_{y=\rho_x}^{y=\sigma_x} dx = 0,\end{aligned} \tag{4}$$

where r_y and s_y are limiting values of x in S for a given y and r and s are the extreme values of x and where ρ_x, σ_x, ρ, σ have like meanings under interchange of rôles of x and y.

If in the first and second integrals in (4) we introduce the variable t in place of the given variable of integration by means of the relations

$$t = \frac{y - \rho}{\sigma - \rho}, \qquad t = \frac{x - r}{s - r},$$

respectively, each integral reduces to an integral with respect to t from

0 to 1, and hence (4) itself may be put in the form

$$\int_0^1 \mathrm{B}\,dt = 0 \tag{5}$$

where B is a sort of bilinear form in the functions u and v and their first derivatives, the arguments of the functions being specialized in the way implied by the transformations involved (so that each of them is a function of a single variable t).

We now require linear homogeneous boundary conditions on u and v separately, independent of λ, such that equation (5) is implied by the boundary conditions alone. We shall presently make clear the nature of these conditions by the explicit treatment of a particular case. For the present we shall suppose that these conditions have been set up in such way that the u-problem and the v-problem have the same characteristic values λ and that these are enumerably infinite in number. This done, we denote the characteristic values λ and the corresponding solutions by λ_i, u_i, v_i, $i = 1, 2, 3, \cdots$.

From (1) and (2) we now have the relations

$$L(u_i) + \lambda_i L_1(u_i) = 0,$$
$$M(v_j) + \lambda_i M_1(v_j) = (\lambda_i - \lambda_j) M_1(v_j).$$

If we multiply the first of these equations member by member by v_j and the second by $-u_i$, add the resulting equations member by member, and then integrate over S, the resulting first member has the value zero on account of the boundary conditions, and hence the second member also has the value zero. Omitting the non-zero factor $\lambda_i - \lambda_j$, this gives us the relation

$$\iint_S u_i M_1(v_j)\,dxdy = 0, \quad i \neq j. \tag{6}$$

Similarly, we have

$$\iint_S v_i L_1(u_j)\,dxdy = 0, \quad i \neq j. \tag{7}$$

These two (equivalent) sets of relations express the generalized biorthogonality conditions for the problem in consideration.

If we suppose that the integral $\iint u_i M_1(v_i)\,dxdy$ taken over S is different from zero for every i we have a ready means of determining formally the coefficients c_k in the formal expansion

$$f(x, y) = \sum_{k=1}^{\infty} c_k u_k(x, y) \tag{8}$$

of a function $f(x, y)$ of two variables in terms of the functions u_k. In fact, we obtain readily the relations

$$
(9) \qquad c_k = \frac{\displaystyle\iint_S f(x, y) M_1(v_k)\,dxdy}{\displaystyle\iint_S u_k(x, y) M_1(v_k)\,dxdy}, \qquad k = 1, 2, \cdots.
$$

In order both to make the method of procedure more explicit and to illustrate the greater simplicity of the self-adjoint case of the foregoing problem let us consider an instance associated with a class of equations of great importance in theoretical physics, namely, equations of the form

$$
(10) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda(g + h)u = 0,
$$

where g is a function of x alone and h is a function of y alone. For the region S we take now the square $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$; and we use the boundary conditions

$$
(11) \qquad \begin{array}{ll} u(t, 1) - u(t, 0) = 0, & u(1, t) - u(0, t) = 0, \\ u_y(t, 1) - u_y(t, 0) = 0, & u_x(1, t) - u_x(0, t) = 0, \end{array}
$$

where the subscript y or x denotes the partial derivative with respect to this variable. It is easy to verify that problem (10), (11) is self-adjoint; that is to say, if one writes the same problem in v one finds that these two problems, one in v and one in u, satisfy those conditions already imposed in the more general problems associated with (1) and (2).

Equation (6) now has the special form

$$
(12) \qquad \int_0^1\int_0^1 (g + h)u_i v_j\,dxdy = 0, \quad i \neq j,
$$

so that for the coefficients c_k in (8) we have the simpler values

$$
(13) \qquad c_k = \frac{\int_0^1\int_0^1 (g + h)u_k(x, y)f(x, y)\,dxdy}{\int_0^1\int_0^1 (g + h)\{u_k(x, y)\}^2\,dxdy}, \qquad k = 1, 2, 3, \cdots.
$$

Instead of considering all the solutions of (10) which satisfy conditions (11) we may fix our attention on any convenient class of them, as for instance those solutions which can be written in the form

$$
(14) \qquad u(x, y) = \rho(x)\sigma(y),
$$

a class of particular solutions of great importance in theoretical and applied mechanics. Substituting $u(x, y)$ of the form (14) into (10) and (11) we find that a constant μ exists such that

$$
(15) \quad \rho''(x) + \lambda g(x)\rho(x) + \mu\rho(x) = 0, \quad \rho(1) - \rho(0) = 0, \quad \rho'(1) - \rho'(0) = 0,
$$

$$
(16) \quad \sigma''(y) + \lambda h(y)\sigma(y) - \mu\sigma(y) = 0, \quad \sigma(1) - \sigma(0) = 0, \quad \sigma'(1) - \sigma'(0) = 0.
$$

Each of the last two differential systems is self-adjoint. Together they form [if written in a single variable x] a system of two differential equations (with boundary conditions) in two parameters λ and μ. General problems of this type have already been treated in § 3. It was through the emergence of the multiple parameter problem in connection with partial differential equations that I was first led to seek the formulation of the general multiple parameter problem for algebraic systems and also for the transcendental cases treated in this memoir.

Let us derive the equations for the characteristic values of λ and μ in equations (15) and (16). Let $\rho_1(x, \lambda, \mu)$ and $\rho_2(x, \lambda, \mu)$ be a fundamental system of solutions of the differential equation in (15) and seek a solution $\rho(x, \lambda, \mu)$ so that the boundary conditions shall be satisfied. Constants c_1 and c_2 are to exist such that

$$\rho(x, \lambda, \mu) = c_1\rho_1(x, \lambda, \mu) + c_2\rho_2(x, \lambda, \mu);$$

hence we have through use of the boundary conditions in (15) the relations

$$\begin{aligned} c_1\{\rho_1(1, \lambda, \mu) - \rho_1(0, \lambda, \mu)\} + c_2\{\rho_2(1, \lambda, \mu) - \rho_2(0, \lambda, \mu)\} &= 0, \\ c_1\{\rho_1'(1, \lambda, \mu) - \rho_1'(0, \lambda, \mu)\} + c_2\{\rho_2'(1, \lambda, \mu) - \rho_2'(0, \lambda, \mu)\} &= 0. \end{aligned}$$

The condition of consistency of these equations is that

$$(17) \qquad \begin{vmatrix} \rho_1(1, \lambda, \mu) - \rho_1(0, \lambda, \mu) & \rho_2(1, \lambda, \mu) - \rho_2(0, \lambda, \mu) \\ \rho_1'(1, \lambda, \mu) - \rho_1'(0, \lambda, \mu) & \rho_2'(1, \lambda, \mu) - \rho_2'(0, \lambda, \mu) \end{vmatrix} = 0,$$

a condition on λ and μ alone. Similarly, if $\sigma_1(y, \lambda, \mu)$ and $\sigma_2(y, \lambda, \mu)$ are a fundamental system of solutions of the differential equation in (16), we are led to the following second condition on λ and μ:

$$(18) \qquad \begin{vmatrix} \sigma_1(1, \lambda, \mu) - \sigma_1(0, \lambda, \mu) & \sigma_2(1, \lambda, \mu) - \sigma_2(0, \lambda, \mu) \\ \sigma_1'(1, \lambda, \mu) - \sigma_1'(0, \lambda, \mu) & \sigma_2'(1, \lambda, \mu) - \sigma_2'(0, \lambda, \mu) \end{vmatrix} = 0.$$

The common solutions of equations (17) and (18) are the characteristic values for problem (15), (16). They are also the characteristic values for problem (10), (11) when the solutions u are restricted to be of the form given in (14).

Let us consider for a moment the special case in which $g = h = 1$. Then equations (15) and (16) have solutions (among others) of the form

$$\rho = e^{\alpha x}, \qquad \sigma = e^{\beta y},$$

where α and β are constants. The boundary conditions then require that we must have

$$e^{\alpha} - 1 = 0, \qquad \alpha(e^{\alpha} - 1) = 0, \qquad e^{\beta} - 1 = 0, \qquad \beta(e^{\beta} - 1) = 0.$$

Hence $\alpha = 2m\pi i$ and $\beta = 2n\pi i$ where m and n are integers, positive or negative or zero. Then corresponding characteristic values λ and μ [perhaps not the complete set] are given by the relations $\lambda + \mu = -4m^2\pi^2$ and $\lambda - \mu = -4n^2\pi^2$, or

$$\lambda = -2\pi^2(m^2 + n^2), \qquad \mu = -2\pi^2(m^2 - n^2),$$

where m and n are integers. The corresponding solutions $u(x, y)$ may now be written

$$u_{mn}(x, y) = e^{2m\pi ix}e^{2n\pi iy}.$$

The expansion of $f(x, y)$ may be written

$$f(x, y) = \sum_{m,\, n=-\infty}^{+\infty} c_{mn}e^{2m\pi ix}e^{2n\pi iy}.$$

For a real-valued function $f(x, y)$ it is easy to transform this into an expansion in sines and cosines of integral multiples of x and y with real coefficients.

Let us consider the extension of our problem to adjoint systems of partial differential equations involving a single parameter. As a typical instance we take the system

$$\begin{aligned} \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} &= \sum_{j=1}^{n}\left(a_{ij}\frac{\partial u_j}{\partial x} + b_{ij}\frac{\partial u_j}{\partial y} + c_{ij}u_j\right) \\ &\quad + \lambda\sum_{j=1}^{n}\left(\alpha_{ij}\frac{\partial u_j}{\partial x} + \beta_{ij}\frac{\partial u_j}{\partial y} + \gamma_{ij}u_j\right), \quad i = 1, 2, \cdots, n, \end{aligned} \tag{19}$$

and its adjoint system

$$\begin{aligned} \frac{\partial^2 v_i}{\partial x^2} + \frac{\partial^2 v_i}{\partial y^2} &= \sum_{j=1}^{n}\left[- a_{ji}\frac{\partial v_j}{\partial x} - b_{ji}\frac{\partial v_j}{\partial y} + \left(c_{ji} - \frac{\partial a_{ji}}{\partial x} - \frac{\partial b_{ji}}{\partial y}\right)v_j\right] \\ &\quad + \lambda\sum_{j=1}^{n}\left[- \alpha_{ji}\frac{\partial v_j}{\partial x} - \beta_{ji}\frac{\partial v_j}{\partial y} + \left(\gamma_{ji} - \frac{\partial \alpha_{ji}}{\partial x} - \frac{\partial \beta_{ji}}{\partial y}\right)v_j\right], \quad i = 1, 2, \cdots, n. \end{aligned} \tag{20}$$

Let us multiply the first of these by v_i and the second by $-u_i$, add the resulting equations member by member, and sum as to i from 1 to n; thus we have

$$\begin{aligned} &\sum_{i=1}^{n}\left[\frac{\partial}{\partial x}\left(v_i\frac{\partial u_i}{\partial x} - u_i\frac{\partial v_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(v_i\frac{\partial u_i}{\partial y} - u_i\frac{\partial v_i}{\partial y}\right)\right] \\ &\quad - \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\frac{\partial}{\partial x}\left\{a_{ij} + \lambda\alpha_{ij}\right\}\cdot u_jv_i + \frac{\partial}{\partial y}\left\{b_{ij} + \lambda\beta_{ij}\right\}\cdot u_jv_i\right] = 0. \end{aligned} \tag{21}$$

After the method employed in the first part of this section we may now obtain from (21) the condition which must be realized as a consequence of linear homogeneous boundary conditions on the u_i and the v_i separately. In (21) we integrate over a two-dimensional region S of the xy-plane, performing the integration with respect to x for those terms affected by the operator $\partial/\partial x$ and with respect to y for those affected by the operator $\partial/\partial y$. In this way we have a relation analogous to (4) above. Just as in the preceding case we introduce a variable t so as to reduce the two integrals to a single integral of the form

$$\int_0^1 \mathrm{B}dt = 0 \tag{22}$$

where B is a sort of bilinear form. The boundary conditions must be set up so as to realize this relation; they may be set up so as to realize the more special relation $\mathrm{B} = 0$.

Without attempting to characterize these boundary conditions in general [suggestions for forming them are afforded by the earlier part of this section and by the treatment of a like problem in § 1] we suppose that they have been chosen so that those on the u_i are linear and homogeneous in the u_i and their first derivatives and so that those on the v_i are linear and homogeneous in the v_i and their first derivatives and that this has been done in such way as to realize condition (22) as a consequence of the boundary conditions. We shall suppose also that the characteristic values λ_1, λ_2, λ_3, $\cdots$ of the u-problem are the same as those for the v-problem and that these values are infinite in number. The solutions corresponding to λ_k we denote by $u_i^{(k)}$, $v_i^{(k)}$.

In (19) let us write λ_k and $u_i^{(k)}$ for λ and u_i; in (20) let us write λ_l and $v_i^{(l)}$ for λ and v_i, where $k \neq l$. Multiplying the first of these resulting equations by $v_i^{(l)}$ and the second by $-u_i^{(k)}$, adding member by member, summing as to i from 1 to n, integrating as to x and y over the region S, simplifying by use of the relation to which (22) reduces when u_i is replaced by $u_i^{(k)}$ and v_i by $v_i^{(l)}$ [a relation which is valid in view of boundary conditions of the sort described], we have a relation which becomes the following when the non-zero factor $\lambda_k - \lambda_l$ is dropped.

$$\int\!\!\int_S \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{-\alpha_{ji}\frac{\partial v_j^{(l)}}{\partial x} - \beta_{ji}\frac{\partial v_j^{(l)}}{\partial y} + \left(\gamma_{ji} - \frac{\partial \alpha_{ji}}{\partial x} - \frac{\partial \beta_{ji}}{\partial y}\right) v_j^{(l)}\right\} u_i^{(k)}\right] dxdy = 0, \quad k \neq l. \tag{23}$$

Similarly we have the equivalent relation

$$\int\!\!\int_S \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\alpha_{ij}\frac{\partial u_j^{(k)}}{\partial x} + \beta_{ij}\frac{\partial u_j^{(k)}}{\partial y} + \gamma_{ij}u_j^{(k)}\right\} v_i^{(l)}\right] dxdy = 0, \quad k \neq l. \tag{24}$$

Now let $f_i(x, y)$, $i = 1, 2, \cdots, n$, be a system of n functions of the two variables x and y and let us seek simultaneous expansions of the form

$$f_i(x, y) = \sum_{\nu=1}^{\infty} c_\nu v_i^{(\nu)}(x, y), \quad i = 1, 2, \cdots, n, \tag{25}$$

where the c_ν are independent of i as well as of x and y. We seek to determine the c_ν on the supposition that the expression in the first member of (24) is different from zero whenever $k = l$. We multiply both members of (25) by

$$\alpha_{ij}\frac{\partial u_j^{(k)}}{\partial x} + \beta_{ij}\frac{\partial u_j^{(k)}}{\partial y} + \gamma_{ij}u_j^{(k)},$$

sum both members as to j from 1 to n and as to i from 1 to n, and integrate in the resulting equation as to x and y over the region S. In view of (24) it is seen that c_k is the only coefficient left in the second member and that we have for c_k the value

$$(26)\quad c_k = \frac{\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\alpha_{ij}\frac{\partial u_j^{(k)}}{\partial x} + \beta_{ij}\frac{\partial u_j^{(k)}}{\partial y} + \gamma_{ij}u_j^{(k)}\right\} f_i(x, y)\,dxdy}{\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\alpha_{ij}\frac{\partial u_j^{(k)}}{\partial x} + \beta_{ij}\frac{\partial u_j^{(k)}}{\partial y} + \gamma_{ij}u_j^{(k)}\right\} v_i^{(k)}(x, y)\,dxdy},$$

$$k = 1, 2, \cdots.$$

In case $\alpha_{ij} \equiv 0 \equiv \beta_{ij}$ relations (24) and (26) reduce to the notably simpler forms

$$\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij}u_j^{(k)}v_i^{(l)}dxdy = 0, \quad k \neq l,$$

$$c_k = \frac{\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij}u_j^{(k)}f_i dxdy}{\int\int_S \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{ij}u_j^{(k)}v_i^{(k)}dxdy}, \quad k = 1, 2, 3, \cdots.$$

Just as the problems in ordinary differential equations with one parameter can be generalized, as we have seen, to the case of any finite number of parameters so the problems treated in the foregoing part of this section can be extended to partial differential equations and systems in any finite number of parameters. Moreover, any finite number of independent variables may be employed in place of the two variables x and y of the foregoing treatment.

If we use the Volterra limiting process for passing from a finite number of differential equations to an integro-differential equation we are led from equations (19) and (20) above to certain adjoint partial integro-differential equations. For the sake of simplicity we shall confine our attention to the case in which the functions a_{ij}, b_{ij}, α_{ij}, β_{ij} are all identically zero. Then the integro-differential equations are the following:

$$(27)\quad \frac{\partial^2 u(x, y, s)}{\partial x^2} + \frac{\partial^2 u(x, y, s)}{\partial y^2} = \int_\alpha^\beta [c(x, y, s, t) + \lambda\gamma(x, y, s, t)]u(x, y, t)dt,$$

$$(28)\quad \frac{\partial^2 v(x, y, s)}{\partial x^2} + \frac{\partial^2 v(x, y, s)}{\partial y^2} = \int_\alpha^\beta [c(x, y, t, s) + \lambda\gamma(x, y, t, s)]v(x, y, t)dt.$$

These equations may be treated by processes closely analogous to those by which (19) and (20) were treated. We summarize the main results. Corresponding to (23) and (24) we now have

$$(29)\qquad \iint_S \left[\int_\alpha^\beta \int_\alpha^\beta \gamma(x, y, t, s) v^{(k)}(x, y, t) u^{(l)}(x, y, s)\, ds\, dt\right] dx\, dy = 0, \quad k \neq l.$$

Then if we seek expansions of the form

$$(30)\qquad f(x, y, s) = \sum_{\nu=1}^{\infty} c_\nu v^{(\nu)}(x, y, s)$$

we have for the coefficients c_k the values

$$(31)\qquad c_k = \frac{\iint_S \left[\int_\alpha^\beta \int_\alpha^\beta \gamma(x, y, s, t) u^{(k)}(x, y, t) f(x, y, s)\, ds\, dt\right] dx\, dy}{\iint_S \left[\int_\alpha^\beta \int_\alpha^\beta \gamma(x, y, s, t) u^{(k)}(x, y, t) v^{(k)}(x, y, s)\, ds\, dt\right] dx\, dy}, \qquad k = 1, 2, \cdots.$$

7. **On a Problem Arising in the Theory of Vibrating Plates.**—An important differential equation in the theory of vibrating plates is the following:

$$(1)\qquad \frac{\partial^4 \xi}{\partial x^4} + 2\frac{\partial^4 \xi}{\partial x^2 \partial y^2} + \frac{\partial^4 \xi}{\partial y^4} + \alpha \frac{\partial^2 \xi}{\partial t^2} = 0.$$

Here α is a constant and ξ is a function of x, y, t. In the theory of vibrating plates one desires to have a solution $\xi(x, y, t)$ of (1) which for $t = 0$ reduces to a given function $f(x, y)$ so that we have $\xi(x, y, 0) = f(x, y)$. Such solutions $\xi(x, y, t)$ have been found in a variety of ways in terms of particular solutions of the form

$$(2)\qquad \xi(x, y, t) = u(x, y)e^{int}$$

where n is a constant. If we substitute this value of ξ into (1) we have for u the equation

$$(3)\qquad L(u) + \lambda u \equiv \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} + \lambda u = 0,$$

where $\lambda = -n^2\alpha$. We shall formulate the expansion problem as connected with equation (3) in the general form* suggested by the standard procedure which we follow under the guidance of the basic algebraic theory which directs our investigation.

The equation adjoint to the equation $L(u) + \lambda u = 0$ is obviously the equation $L(v) + \lambda(v) = 0$. We have the following readily verified basic identity:

---

* A number of special cases of this problem have been treated (with important results) in connection with the usual theory of vibrating plates.

$$
\begin{aligned}
vL(u) - uL(v) \equiv {} & \frac{\partial}{\partial x}\left[v\frac{\partial^3 u}{\partial x^3} - u\frac{\partial^3 v}{\partial x^3} - \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial x^2} \right. \\
& + sv\frac{\partial^3 u}{\partial x \partial y^2} + (s-2)u\frac{\partial^3 v}{\partial x \partial y^2} - s\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2} \\
& \left. - (s-2)\frac{\partial u}{\partial x}\frac{\partial^2 v}{\partial y^2} \right] + \frac{\partial}{\partial y}\left[v\frac{\partial^3 u}{\partial y^3} - u\frac{\partial^3 v}{\partial y^3} \right. \\
& - \frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + (2-s)v\frac{\partial^3 u}{\partial x^2 \partial y} \\
& \left. - su\frac{\partial^3 v}{\partial x^2 \partial y} + (s-2)\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x^2} + s\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2} \right].
\end{aligned}
\tag{4}
$$

In this identity s denotes an arbitrary parameter.

Let S be any closed region of the xy-plane whose boundary is cut twice and only twice* by a line parallel to either axis and passing through the region. Then if u and v are two solutions of (3) we have

$$
0 = \iint_S [vL(u) - uL(v)]\,dxdy = \iint_S \left\{ \frac{\partial}{\partial x}\mathrm{B}_1 + \frac{\partial}{\partial y}\mathrm{B}_2 \right\} dxdy,
$$

where B_1 and B_2 denote the first and second brackets expressions in the second member of (4). From this we have readily the relation

$$
\int_\rho^\sigma [\mathrm{B}_1]_{x=r_y}^{x=s_y} dy + \int_r^s [\mathrm{B}_2]_{y=\rho_x}^{y=\sigma_x} dx = 0 \tag{5}
$$

where r_y, s_y, r, s, ρ_x, σ_x, ρ, σ have the same meanings as in connection with equation (4) of § 6. By the same transformation as is employed in § 6 we can reduce relation (5) to the form

$$
\int_0^1 \mathrm{B}\,dt = 0 \tag{6}
$$

where B is a sort in bilinear form in the functions u and v and their derivatives up to the third, the arguments of the functions being specialized in the way implied by the transformations involved (so that each of the functions is a function of the single variable t).

We now require (as usual) linear homogeneous boundary conditions on u and v separately such that equation (6) is implied by the boundary conditions alone. A particular class of cases of great importance is that in which the boundary conditions imply the stronger relation $\mathrm{B} \equiv 0$. Leaving off for the moment the actual formulation of these boundary conditions let us suppose that they have been set up in such a way that the problem for the equation $L(u) + \lambda u = 0$ and the u-conditions has the same characteristic values λ as the problem for the equation $L(v) + \lambda v = 0$ and the v-conditions and that these characteristic values are enumerably infinite in number. We denote the characteristic values λ and the corresponding solutions by $\lambda_i, u_i, v_i, i = 1, 2, 3, \cdots$. Since equation (3) is self-adjoint it is seen that $u_i = v_i$ for every i provided that the u-conditions and the v-conditions are the same. In this case the entire u-problem is self-adjoint.

---

* The reader will see what modifications of the treatment are sufficient to deal with the case when these lines cut the boundary any finite number of times.

Now we have the relations

$$L(u_i) + \lambda_i u_i = 0, \qquad L(v_j) + \lambda_i v_j = (\lambda_i - \lambda_j) v_j, \quad i \neq j.$$

Let us multiply the first of these equations by v_j and the second by $-u_i$, add the resulting equations member by member, and then integrate over S; the resulting first member has the value zero on account of the boundary conditions, and hence the second member also has the value zero. Omitting the non-zero factor $\lambda_i - \lambda_j$, this gives us the relation

$$\iint_S u_i v_j \, dxdy = 0, \quad i \neq j. \tag{7}$$

For the self-adjoint case this reduces to

$$\iint_S u_i u_j \, dxdy = 0, \quad i \neq j. \tag{7$_1$}$$

If we suppose the integral in the first member of (7) to be different from zero when $j = i$ (as it certainly is in the self-adjoint case) we have a ready means of determining the coefficients c_k in the formal expansion

$$f(x, y) = \sum_{k=1}^{\infty} c_k u_k(x, y) \tag{8}$$

of a function $f(x, y)$ of two variables in terms of the functions u_k. In fact we obtain readily the relations

$$c_k = \frac{\iint_S f(x, y) v_k(x, y) \, dxdy}{\iint_S u_k(x, y) v_k(x, y) \, dxdy}, \quad k = 1, 2, \cdots. \tag{9}$$

In the self-adjoint case we have the simpler expressions

$$c_k = \frac{\iint_S f(x, y) u_k(x, y) \, dxdy}{\iint_S \{u_k(x, y)\}^2 \, dxdy}, \quad k = 1, 2, \cdots. \tag{9$_1$}$$

Suppose that we desire a solution $\xi(x, y, t)$ of (1) which for $t = 0$ reduces to $f(x, y)$ so that we have formally

$$\xi(x, y, 0) = \sum_{k=1}^{\infty} c_k u_k(x, y),$$

where the c_k have the values just obtained. Such a solution is formally the following:

$$\xi(x, y, t) = \sum_{k=1}^{\infty} c_k u_k(x, y) e^{\sqrt{\lambda_k/a}\, t}$$

This is the actual solution desired provided that the boundary conditions are such as to render the solution unique and provided that the requisite conditions of convergence are satisfied. We have thus completed the formulation of the expansion problem for the present case.

It is desirable to observe how easily one can extend the formal theory associated with equations (3) to (9) inclusive to that depending on the equation $L(u) + \lambda g u = 0$ in place of equation (3), where g is a function of x and y. The boundary conditions will be set up exactly as before, the function g not intervening in this part of the discussion. In the paragraph containing (7) we shall have a slight modification of formulæ and there will result the generalized biorthogonality conditions

$$\iint_S g u_i v_j dxdy = 0, \quad i \neq j;$$

and these give rise to the relations

$$c_k = \frac{\iint_S g(x, y) f(x, y) v_k(x, y) dxdy}{\iint_S g(x, y) u_k(x, y) v_k(x, y) dxdy}, \quad k = 1, 2, 3, \cdots,$$

for determining the coefficients c_k in the formal expansion (8).

It is clear that the formal expansion theory connected with the equation $L(u) + \lambda g u = 0$ can be readily carried over to the system

$$L(u^{(i)}) = \sum_{j=1}^{n} (a_{ij} u^{(j)} + \lambda g_{ij} u^{(j)}), \quad i = 1, 2, \cdots, n,$$

and to the simultaneous expansion of n functions in terms of the solutions obtained. This in turn can be made to yield heuristically the formulation of a corresponding problem in partial integro-differential equations. These two problems in one parameter can thence be easily extended to problems in any finite number of parameters. The method of making these extensions is sufficiently obvious in view of the methods already employed in this paper in closely related problems.

In order to make clear the nature of the boundary conditions associated with equation (3) and described above through the intervention of equation (6), let us set them up explicitly and generally for the case when S is a rectangle and relation (6) is to be satisfied by requiring that B shall be identically equal to zero. We define S as that rectangle in which $a \leqq x \leqq b$, $c \leqq y \leqq d$.

Let us now write more explicitly

$$B_1(x, y), \qquad B_2(x, y), \tag{10}$$

respectively, for the forms in the first and second brackets expressions in relation (4). Then for B we now have the value

$$\begin{aligned} B = (d - c)\{B_1(b, c + (d - c)t) - B_1(a, c + (d - c)t)\} \\ + (b - a)\{B_2(a + (b - a)t, d) - B_2(a + (b - a)t, c)\}. \end{aligned} \tag{11}$$

Now let us think of $B_1(x, y)$ as a sort of bilinear form in the two sets of variables

$$u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^3 u}{\partial x^3}, \frac{\partial^3 u}{\partial x \partial y^2} \quad \text{and} \quad v, \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 v}{\partial y^2}, \frac{\partial^3 v}{\partial x^3}, \frac{\partial^3 v}{\partial x \partial y^2}. \tag{12}$$

Let us form the matrix $\|a_{ij}\|$ of this bilinear form where the element a_{ij} of the ith column and jth row is the coefficient of the product of the ith function in the first set by the jth function in the second set. Then we have

$$\|a_{ij}\| \equiv \begin{Vmatrix} 0 & 0 & 0 & 0 & 1 & s \\ 0 & 0 & -1 & -s & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 - s & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ s - 2 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}.$$

It is easy to show that this matrix is of rank 4; for the last row may be made to consist of elements zero by adding to it $(s - 2)$ times the second last row, the fourth row may be made to consist of elements zero by adding to it $(s - 2)$ times the third row, the last column may be made to consist of elements zero by subtracting from it s times the second last column, and the fourth column may be made to consist of elements zero by subtracting from it s times the third column, the matrix so obtained being the following (which obviously is of rank 4):

$$\begin{Vmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix}$$

In $B_1(x, y)$ we have the six variables u and the six variables v listed in (12). In $B_2(x, y)$ we have similar six variables u and six variables v. The matrix of the bilinear form $B_2(x, y)$ is of rank 4, as one may readily verify by the method used for $B_1(x, y)$. Hence we may look upon B, defined as in (11), as a bilinear function of 24 variables u and 24 variables v, and this bilinear form is of rank 16. Hence B may be put into the form

$$\mathrm{B} = \sum_{i=1}^{16} U_i(u)V_i(v), \tag{13}$$

where for each i $U_i(u)$ is a linear homogeneous function of the 24 variables u and $V_i(v)$ is a linear homogeneous function of the 24 variables v. The set $U_i(u)$, $i = 1, 2, \cdots, 16$, is linearly independent, and so also is the set $V_i(v)$, $i = 1, 2, \cdots, 16$. Then the boundary conditions which we take for the u-problem are

$$U_i(u) = 0, \quad i = 1, 2, \cdots, 8, \tag{14}$$

while those for the v-problem are

$$V_i(v) = 0, \quad i = 9, 10, \cdots, 16. \tag{15}$$

[For some parts of the theory one may take any number k of conditions $U_i(u) = 0$ not greater than 15, $i = 1, 2, \cdots, k$, and then use for the v-conditions $V_i(v) = 0, i = k + 1, \cdots, 16$. But the form of conditions given in (14) and (15) is needful for certain parts of the development.] If the conditions on v in (15) are the same as those on u in (14) the problem is self-adjoint and the maximum of elegance is secured.

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